# COMPUTATION OF CONNECTION MATRICES USING THE SOFTWARE PACKAGE `conley`

MOHAMED BARAKAT AND STANISLAUS MAIER-PAAPE

ABSTRACT. In this paper we demonstrate the power of the computer algebra package `conley`, which enables one to compute connection and transition matrices, two of the main algebraic tools of the CONLEY index theory. In particular, we study the CAHN-HILLIARD equation on the unit square and extend the results obtained in [Maier-Paape et al., 2007] to a bigger range of the bifurcation parameter. Besides providing several explicit computations using `conley`, the definition of connection matrices is reconsidered, simplified, and presented in a self-contained manner in the language of CONLEY index theory. Furthermore, we introduce so-called energy induced bifurcation intervals, which can be utilized by `conley` to differential equations with a parameter. These bifurcation intervals are used to automatically path-follow the set of connection matrices at bifurcation points of the underlying set of equilibria.

CONTENTS

## 1. Introduction

In recent years Conley index theory has become a very powerful tool for studying invariant sets of dynamical systems (see for example [Mischaikow and Mrozek, 2002]). Connection matrices, a central concept in that theory, enables one to investigate and prove the existence of heteroclinic connections between (isolated) invariant sets.

In the dynamical systems community connection matrices do have a touch of mystery. The reason is twofold. On the one hand, the concept heavily relies on homological algebra and even elementary aspects like the definition or the existence of connection matrices [Franzosa, 1989] are difficult to comprehend without a solid algebraic background. On the other hand, the full power of the connection matrix concept can only be reached, when non-trivial intrinsic dynamical arguments are coupled with the algebraic arguments (see for instance [Maier-Paape et al., 2007]).

This is where our contribution here comes into play. Our main purpose is to automatize the above mentioned algebraic (and to some extent also the dynamical) arguments in order to be able to calculate (even larger) connection matrices with a computer. Since many relevant examples arise from bifurcations, where the internal structure of the isolated invariant set under consideration gets more and more complicated, we also put emphasis on the possibility to bridge over bifurcation points. This is mainly worked out in Section 5, where we revisit and extend the results obtained in [Maier-Paape et al., 2007] for the global attractor of the Cahn-Hilliard equation on a square. Due to computer power we were not only able to verify the results in [Maier-Paape et al., 2007], but also to rebase the proofs on more conventional arguments, which finally allowed us to push the investigation of the bifurcation diagram to a new parameter range. We also introduce in Definition 5.1 the new algebraic concept of a *energy induced bifurcation interval*, which is very helpful to automatically path-follow connection matrices at bifurcation points. Although clearly not yet there, in the long run it now seems possible to develop software autonomously analyzes the fine structure of dynamical systems.

Although the concept of connection matrices naturally leads to braids as already proposed in [Franzosa, 1989] we show, following [Barakat and Robertz, 2009], that exactly the same concept is obtained by only imposing isomorphisms of long exact sequences. Clearly this makes the verification of connection matrices a lot easier. In contrast to [Barakat and Robertz, 2009], which used a purely algebraic setup, we here give the simplified definition of connection matrices within the conventional language of Conley index theory and with a minimum of algebraic preliminaries. Since our primary motivation is the application to dynamical systems we try to keep the obviously necessary algebra part as small as possible. Nevertheless we introduce enough Conley index theory to keep the definition of connection matrices self–contained. We therefore in Section 3 reproduce a minimum of the basic notations and give the braids free definition of connection matrices in Definition 3.5. Before doing so, in Section 2 we give a few simple examples illustrating the intuitive and geometric ideas behind CONLEY index theory without any formal definitions. These examples give a preview of the nature of results that can be obtained and furthermore show how CONLEY's ideas in fact generalize the classical MORSE theory.

In order to introduce the syntax of the package `conley`, we discuss in Section 4 two easy examples of FRANZOSA ([Franzosa, 1989]) on the non–uniqueness of connection matrices and the corresponding transition matrix.

The two main procedures of the software package `conley` are `ConnectionMatrices` and `TransitionMatricesGenerators`. They require as their input the relevant MORSE set together with an admissible ordering, along with CONLEY index data of isolated invariant sets. An elaborate version of the examples of Sections 4 and 5 can be found in form of Maple worksheets in the library of examples on the homepage of the `conley` project [Barakat et al., 2008].

In future we plan to port the Maple package `conley` to GAP4 as part of the emerging new `homalg` project (`http://homalg.math.rwth-aachen.de/`). Within GAP4, using its object oriented programming philosophy, the input syntax will be considerably simplified. The new `homalg` project allows the combination of several efficient software resources under one hat, whereby computation time of the `conley` procedures will be cut by a huge factor. In particular, it would also be possible to combine algebraic together with numerical software for bifurcation analysis.

## 2. Historical Remarks

In order to give some examples illustrating the usefulness of CONLEY index theory we first make some historical remarks. This introductory section is not necessary to understand the formal theory starting in Section 3.

One of the motivations of CONLEY was to generalize MORSE theory. The main idea of MORSE theory is to study the topological properties of an $n$-dimensional smooth manifold $X$ by studying the critical points of a so-called MORSE function $f : X \to \mathbb{R}$. A MORSE function is a smooth function with non-degenerate critical points. For such a non-degenerate critical point $p \in X$ the MORSE lemma guarantees the existence of a coordinate system $x = (x_1, \dots, x_n)$ around $p$ such that $f$ can be written as

$$f(x) = f(p) - x_1^2 - \cdots - x_\gamma^2 + x_{\gamma+1}^2 + \cdots + x_n^2.$$

The number $\gamma$ is called the MORSE index of the critical point $p$ and denoted by index($p$).

Let $c_\gamma$ denote the number of critical points of MORSE index $\gamma$. The MORSE formula

$$\chi(X) = \sum_{\gamma=0}^{n} (-1)^\gamma c_\gamma. \tag{1}$$

computes the EULER-POINCARÉ characteristic $\chi(X)$ in terms of the indices of the critical points of $f$, where by definition

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \beta_i(X),$$

which is the alternating sum of the Betti numbers.

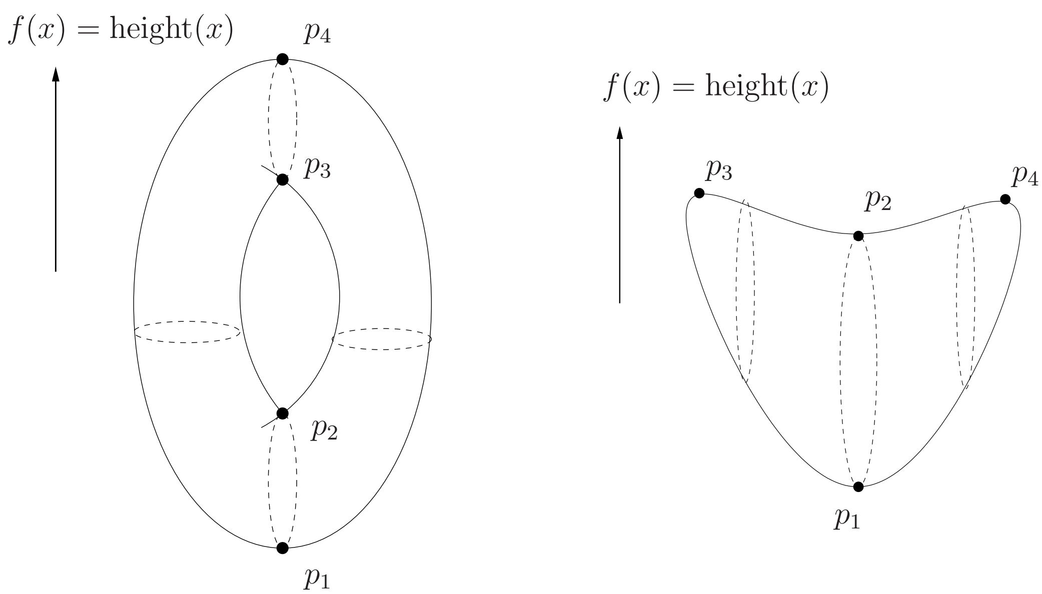



Figure 1. Critical points on torus $\mathbb{T}$ and heart $\mathbb{H}$

We illustrate this with two examples in Figure 1. It turns out that the height function $f(x) = \text{height}(x)$ is a Morse function for both surfaces. On the torus $\mathbb{T}$ we have four critical points $p_1, \dots, p_4$ with $c_0 = c_2 = 1$ and $c_1 = 2$. On the heart $\mathbb{H}$ we have also four critical points but $c_0 = c_1 = 1$ and $c_2 = 2$. Hence $\chi(\mathbb{T}) = 0$ and $\chi(\mathbb{H}) = 2$.

Now Conley's idea was to replace $f$ by the flow generated by the gradient $\nabla f$ and develop a general index theory for flows on manifolds. Note that in Conley's theory the flow is not necessarily a gradient flow.

The first central notion in Conley index theory is the Conley index of isolated invariant sets (a formal definition will be given in (4)). For the purpose of this introductory section we only need the Conley index of the whole manifold $X$ and those of the hyperbolic equilibria $p$ of the flow, which correspond to the (non-degenerate) critical points of the Morse function $f$.

Define the *homology* Conley *index* $CH_*(X)$ of the closed manifold $X$ as the graded object of homology groups of $X$, i.e.

$$CH_*(X) := H_*(X) = H_*(X, \emptyset) = (H_0(X), H_1(X), \dots). \tag{2}$$

For the purpose of this section we take homology with values in a field $\mathbb{K}$ for simplicity.

Let $\gamma$ denote the dimension of the unstable manifold of a hyperbolic equilibrium $p \in X$. If the flow $\varphi$ is a gradient flow of a Morse function $f$, then $\gamma$ is the Morse index index$(p)$ mentioned above. The homology Conley index $CH_*(p)$ of $p$ now generalizes the Morse index in the following way (see also Prop. 3.1): It is again a graded object of $\mathbb{K}$-vector spaces $CH_*(p) = (CH_0(p), CH_1(p), \ldots)$ such that

$$CH_i(p) \cong \begin{cases} \mathbb{K} & \text{if } i = \gamma \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

The second central notion in CONLEY index theory is that of a *connection matrix*.

We assume for simplicity that $\varphi$ is a strict gradient flow with a finite set $P$ of equilibria, all of them hyperbolic. Define the sum of graded objects

$$C = (C_i)_{i \geq 0} = \bigoplus_{p \in P} CH_*(p).$$

Consider a sequence of maps $\Delta = (\Delta_1, \Delta_2, \ldots)$ with $\Delta_i : C_i \to C_{i-1}$ such that $\Delta_{i-1} \circ \Delta_i = 0$, turning $C$ into a complex. We will write $C^\Delta$ for the complex $C$ endowed with $\Delta$ as a boundary operator. A sequence $\Delta$ is called a connection matrix if, among other things (see Def. 3.5), the following property holds:

$$H_i(C^\Delta) \cong CH_i(X). \tag{3}$$

In our heart example from above it can be verified that $\Delta = (\Delta_1, \Delta_2)$ in

$$C^\Delta : 0 \leftarrow \mathbb{K}^1 \xleftarrow{\Delta_1 = \begin{pmatrix} 0 \end{pmatrix}} \mathbb{K}^1 \xleftarrow{\Delta_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{K}^2 \leftarrow 0$$

is a connection matrix.

One of the main results of CONLEY index theory implies that non-trivial entries in this connection matrix correspond to *heteroclinic connections* between $p_3 \to p_2$ and $p_4 \to p_2$. Another major result is FRANZOSA's existence result of a connection matrix [Franzosa, 1989, Thm. 3.8] yielding in particular (3), whereas uniqueness in not always guaranteed.

Therefore, in general, connection matrices may be used to reduce the huge amount of possible heteroclinic connections, and even prove existence of some of the connections.

As a nice application of the above developed notion, the MORSE formula (1) now immediately follows from the existence of a connection matrix. In case $\varphi$ is the gradient flow of a MORSE function $f$ then $\dim_{\mathbb{K}} C_i^\Delta = c_i$, the number of critical points of $f$ with MORSE index $i$. Then

$$\begin{aligned} \chi(X) &= \textstyle\sum_{i=0}^n (-1)^i \dim_{\mathbb{K}} H_i(X) \overset{(2)}{=} \sum_{i=0}^n (-1)^i \dim_{\mathbb{K}} CH_i(X) \\ &\overset{(3)}{=} \textstyle\sum_{i=0}^n (-1)^i \dim_{\mathbb{K}} H_i(C^\Delta) = \sum_{i=0}^n (-1)^i \dim_{\mathbb{K}} C_i^\Delta \\ &= \textstyle\sum_{i=0}^n (-1)^i c_i, \end{aligned}$$

where the fore-last equation is a standard application of the homomorphism theorem.

## 3. CONLEY INDEX THEORY

Let $X$ be a locally compact metric space. The object of study is a *flow* $\varphi : \mathbb{R} \times X \to X$, i.e. a continuous map $\mathbb{R} \times X \to X$ which satisfies $\varphi(0, x) = x$ and $\varphi(s, \varphi(t, x)) = \varphi(s+t, x)$ for all $x \in X$ and $s, t \in \mathbb{R}$. $(X, \varphi)$ is called a *dynamical system*.

3.1. **Homology** CONLEY **index.** The following theory has been initiated by CONLEY [Conley, 1978] in order to study invariant sets of dynamical systems. For a subset $Y \subset X$ define

$$\mathrm{Inv}(Y) := \mathrm{Inv}(Y, \varphi) := \{x \in Y \mid \varphi(\mathbb{R}, x) \subset Y\} \subset Y,$$

the *invariant subset* of $Y$.

A subset $S \subset X$ is *invariant* under the flow $\varphi$, if $S = \mathrm{Inv}(S)$. $S$ is called an *isolated invariant set* if there exists a *compact* set $Y \subset X$ (an *isolating neighborhood*) such that

$$S = \mathrm{Inv}(Y) \subset \mathring{Y},$$

where $\mathring{Y}$ denotes the interior of $Y$.

Let $M$ be an isolated invariant set. A pair of compact sets $(N, L)$ with $L \subset N$ is called an *index pair* for $M$ (cf. [Mischaikow and Mrozek, 2002, Def. 2.4]) if

(1) $\overline{N \setminus L}$ is an *isolating neighborhood* of $M$.
(2) $L$ is *positively invariant*, i.e. $\varphi([0,t], x) \subset L$ for all $x \in L$ satisfying $\varphi([0,t], x) \subset N$.
(3) $L$ is an *exit set* for $N$, i.e. for all $x \in N$ and all $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, there exists a $t_0 \in [0, t_1]$ for which $\varphi([0, t_0], x) \subset N$ and $\varphi(t_0, x) \in L$.

Let $M \subset S$ be an isolated invariant set with index pair $(N, L)$. We associate to such a pair a complex $\mathcal{C}_*(N, L) \cong \mathcal{C}_*(N)/\mathcal{C}_*(L)$ of relative (simplicial or singular ...) chains. The *homology* CONLEY *index* of $M$ is defined by

$$CH_*(M) = H_*(N, L) := H_*(\mathcal{C}_*(N, L)), \tag{4}$$

where $H_*(N, L) = (H_k(N, L))_{k \in \mathbb{Z}_{\geq 0}}$ denotes the relative homology groups (cf. [Mischaikow and Mrozek, 2002, Def. 3.7, Thm. 3.8]). Note, that there always exits an index pair $(N, L)$, such that $H_*(N, L) = H_*(N/L, [L])$ (see [Mischaikow and Mrozek, 2002, Remark 3.9]). We usually take coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Before we proceed, let us recall the homology CONLEY index of some specific isolated invariant sets.

**Proposition 3.1.** *Assume that $S$ contains only a hyperbolic fixed point with an unstable manifold of dimension $n$ (i.e.* MORSE *index $n$). Then $S$ is an isolated invariant set and*

$$CH_k(S) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

*For the remainder of this paper, we will abbreviate this statement by saying that the* CONLEY *index of $S$ is equal to $\Sigma^n$, i.e. write $CH_*(S) = \Sigma^n$.*

Note, that usually $\Sigma^n = (S^n, *)$ denotes the *homotopy* type of the index pair $(N, L)$ of a hyperbolic fixed point of MORSE index $n$. But since we are only interested in homology, we abuse the notation.

In order to apply CONLEY's theory to MORSE decompositions of the attractor, we need to know the CONLEY index of the attractor itself. By the continuation property [Mischaikow and Mrozek, 2002, Thm. 3.10] it is the same as the one of a stable fixed point (see [McCord and Mischaikow, 1996, Prop. 4.1]).

**Proposition 3.2.** *If the dynamical system $(X, \varphi)$ possesses a global attractor $\mathcal{A}$, then we have*

$$CH_k(\mathcal{A}) = \begin{cases} \mathbb{Z}_2 & \textit{for} \quad k = 0\,, \\ 0 & \textit{for} \quad k \neq 0\,. \end{cases}$$

The empty set $S$ is also an isolated invariant set having the trivial Conley index

$$CH_k(S) = 0 \quad \text{for all } k. \tag{5}$$

This occurs e.g. if an isolating neighborhood of a parallel flow is considered. Similarly, a heteroclinic connection between two hyperbolic fixed points stemming from a saddle node bifurcation has trivial Conley index, although the isolated invariant set is no longer empty.

3.2. **Posets.** A set $P$ together with a strict partial order $>$ (i.e. an irreflexive and transitive relation $> \subset P \times P$) is called a *poset* (i.e. partially ordered set) and is denoted by $(P, >)$.

A subset $I \subset P$ is called an *interval* in $(P, >)$ if for all $p, q \in I$ and $r \in P$ the following implication holds:

$$q > r > p \quad \Rightarrow \quad r \in I.$$

The set of all intervals in $(P, >)$ is denoted by $\mathcal{I}(P, >)$.

An $n$-tuple $(I_1, \ldots, I_n)$, $n \geq 2$, of intervals in $(P, >)$ is called *adjacent* if these intervals are mutually disjoint, $\bigcup_{i=1}^n I_i$ is an interval in $(P, >)$ and for all $p \in I_j$, $q \in I_k$ the following implication holds:

$$j < k \quad \Rightarrow \quad p \not> q.$$

The set of all adjacent $n$-tuples of intervals in $(P, >)$ is denoted by $\mathcal{I}_n(P, >)$.

If $(I_1, \ldots, I_n)$ is an adjacent $n$-tuple of intervals in $(P, >)$, then denote $I_1 I_2 \ldots I_n := \bigcup_{i=1}^n I_i$, which by definition is again an interval.

If $(I, J) \in \mathcal{I}_2(P, >)$ as well as $(J, I) \in \mathcal{I}_2(P, >)$, then $I$ and $J$ are said to be *incomparable*.

3.3. Morse **decomposition.** For a subset $Y \subset X$ the *$\omega$-limit set* of $Y$ is $\omega(Y) := \bigcap_{t>0} \overline{\varphi([t, \infty), Y)}$, while the *$\alpha$-limit set* of $Y$ is $\alpha(Y) := \bigcap_{t>0} \overline{\varphi((-\infty, -t), Y)}$.

For two subsets $Y_1, Y_2 \subset X$ define the *set of connecting orbits*

$$\mathrm{Con}(Y_1, Y_2) = \mathrm{Con}(Y_1, Y_2; X) := \{x \in X \mid \alpha(x) \subset Y_1 \text{ and } \omega(x) \subset Y_2\}.$$

Let $S$ be an isolated invariant set and $(P, >)$ be a poset. A finite collection

$$\mathcal{M}(S) = \{M(p) \mid p \in P\}$$

of disjoint isolated invariant subsets $M(p)$ of $S$ is called a Morse *decomposition* if there exists a strict partial order $>$ on $P$, such that for every $x \in S \setminus \bigcup_{p \in P} M(p)$ there exist $p, q \in P$, such that $q > p$ and $x \in \mathrm{Con}(M(q), M(p))$.

The sets $M(p)$ are called Morse *sets*. A partial order on $P$ satisfying this property is said to be *admissible*.

There is a partial order $>_\varphi$ induced by the flow, generated by the relations $q >_\varphi p$ whenever $\mathrm{Con}(M(q), M(p)) \neq \emptyset$. This so called *flow-induced order* is a subset of every admissible order, and in this sense minimal. Normally this order is not known and one is content with a coarser order. If, for example, a Lyapunov or energy function $E$ is known with $E(x) > E(\varphi(t,x))$ for all $t > 0$, whenever $x \in X$ is not a steady state, then defining the partial order $>_E$ by

$$q >_E p, \text{ iff } E(y) > E(x) \text{ for all } y \in M(q) \text{ and } x \in M(p),$$

yields an admissible order, in case the energy levels of all non-equilibrium Morse sets are isolated in the energy spectrum. This order is called *energy-induced order*.

For an interval $I$ define the set

$$M(I) := \bigcup_{p \in I} M(p) \cup \bigcup_{p,q \in I} \operatorname{Con}(M(q), M(p)).$$

$M(I)$ is again an isolated invariant set (cf. [Mischaikow and Mrozek, 2002, Prop. 2.12]). If $(I,J) \in \mathcal{I}_2(P,>)$, then $(M(I), M(J))$ is an *attractor-repeller pair* in $M(IJ)$ (cf. [Mischaikow and Mrozek, 20 Def. 2.1]).

3.4. **Connection matrices.** We start by revising the definition of connection matrices following [Barakat and Robertz, 2009]. Contrary to [Barakat and Robertz, 2009] we apply matrices from the left and hence use the column convention, as widely used in the Conley index literature.

Let $\mathcal{M}(S) = \{M(p) \mid p \in (P,>)\}$ be a Morse decomposition of $S$. Hence each $M(p)$ is an isolated invariant set and the Conley index $CH_*(M(p))$ is well-defined (by (4)). In what follows, we consider the collection $C := \{CH_*(M(p)) \mid p \in P\}$ of abelian groups, which are indexed by $P$, and a group homomorphism

$$\Delta : \bigoplus_{p \in P} CH_*(M(p)) \to \bigoplus_{p \in P} CH_*(M(p)). \tag{6}$$

For an interval $I$ in $(P,>)$ set $C_*(I) := \bigoplus_{p \in I} CH_*(M(p))$ and denote by $\Delta(I) : C_*(I) \to C_*(I)$ the homomorphism $\pi_I \circ \Delta \circ \iota_I$, where $\iota_I : C_*(I) \to C_*(P)$ is the canonical injection and $\pi_I : C_*(P) \to C_*(I)$ is the canonical projection.

If $p_1, p_2 \in P$, we refer to the restriction of $\Delta$ to $C_*(p_2)$ by $\Delta(\cdot, p_2) : C_*(p_2) \to C_*(P)$, and the composition $\pi_{p_1} \circ \Delta(\cdot, p_2)$, where $\pi_{p_1}$ is the projection $C_*(P) \to C_*(p_1)$, is denoted by $\Delta(p_1, p_2) : C_*(p_2) \to C_*(p_1)$. Then $\Delta$ can be visualized as a matrix with $\Delta(\cdot, p_2)$ as its $p_2$-th column and $\Delta(p_1, p_2)$ as its entry at position $(p_1, p_2)$. In particular, for $I \in \mathcal{I}(P,>)$ the homomorphism $\Delta(I)$ may be be represented as

$$\Delta(I) = (\Delta(p_1,p_2))_{p_1,p_2 \in I} : \bigoplus_{p \in I} CH_*(M(p)) \to \bigoplus_{p \in I} CH_*(M(p)).$$

**Definition 3.3** ([Franzosa, 1988, Def. 1.3])**.** $\Delta$ *being as above:*

1. $\Delta$ *is said to be* upper triangular *if* $\Delta(p_1,p_2) \neq 0$ *implies* $p_2 > p_1$ *or* $p_1 = p_2$.
2. $\Delta$ *is said to be* strictly upper triangular *if* $\Delta(p_1,p_2) \neq 0$ *implies* $p_2 > p_1$.
3. $\Delta$ *is called a* boundary map *if it is a homomorphism of degree* $-1$*, i.e. it maps* $C_n(P)$ *to* $C_{n-1}(P)$*, and* $\Delta \circ \Delta = 0$.

**Proposition 3.4** ([Franzosa, 1989, Prop. 3.3])**.** *Let* $C = \{CH_*(M(p)) \mid p \in P\}$ *be as above and let* $\Delta : \bigoplus_{p \in P} CH_*(M(p)) \to \bigoplus_{p \in P} CH_*(M(p))$ *be an upper triangular boundary map. Then:*

(1) *$C_*(I)$ and $\Delta(I)$ form a chain complex denoted by $C_*^\Delta(I)$ for all $I \in \mathcal{I}(P,>)$.*
(2) *For all $(I,J) \in \mathcal{I}_2(P,>)$, the obvious injection and projection maps $i(I,IJ)$ and $p(IJ,J)$ are chain maps and*

$$0 \to C_*^\Delta(I) \xrightarrow{i(I,IJ)} C_*^\Delta(IJ) \xrightarrow{p(IJ,J)} C_*^\Delta(J) \to 0 \tag{7}$$

*is a short exact sequence.*

In other words, the degree $-1$ property and $\Delta\circ\Delta = 0$ endow $C_*(P)$ with a chain complex structure (called $C_*^\Delta(P)$). The property "upper triangular" guarantees that $\Delta(I)$ is also a boundary operator on $C_*(I)$ leading to $C_*^\Delta(I)$. It further implies for a pair $(I,J)$ of adjacent intervals that $\Delta(IJ)_{|C_*(I)} = \Delta(I)$, allowing one to view $C_*^\Delta(I)$ as a subcomplex of $C_*^\Delta(IJ)$, with $C_*^\Delta(J)$ being naturally isomorphic to the quotient complex $C_*^\Delta(IJ)/C_*^\Delta(I)$, making (7) a short exact sequence.

The first statement of Proposition 3.4 allows one to define the homology groups

$$H_*(C_*^\Delta(I)) := \ker \Delta(I)/\mathrm{im}\Delta(I),$$

shortly denoted as $H_*\Delta(I)$, while the second statement leads for each $(I,J) \in \mathcal{I}_2(P,>)$ to a long exact homology sequence

$$\cdots \to H_n\Delta(I) \to H_n\Delta(IJ) \to H_n\Delta(J) \xrightarrow{\delta_n} H_{n-1}\Delta(I) \to \cdots, \tag{8}$$

where $\delta_*$ are the connecting homomorphisms constructed by the snake Lemma.

To state the definition of a connection matrix we still need some more preliminaries from the dynamics side. For a pair $(I,J)$ of adjacent intervals $(M(I), M(J))$ is an attractor-repeller pair for the isolated invariant set $M(IJ)$, as stated before.

By definition an *index triple* $(N_2, N_1, N_0)$ for the attractor-repeller pair $(M(I), M(J))$ satisfies $N_0 \subset N_1 \subset N_2$ and

(i) $(N_2, N_0)$ is an index pair for the isolated invariant set $M(IJ)$;
(ii) $(N_2, N_1)$ is an index pair for the repeller $M(J)$;
(iii) $(N_1, N_0)$ is an index pair for the attractor $M(I)$.

The existence of an index triple $(N_2, N_1, N_0)$ for the attractor-repeller pair $(M(I), M(J))$ is always guaranteed (cf. [Mischaikow and Mrozek, 2002, Thm. 4.2]), providing a short exact sequence of chain complexes

$$0 \to \mathcal{C}_*(N_1, N_0) \to \mathcal{C}_*(N_2, N_0) \to \mathcal{C}_*(N_2, N_1) \to 0,$$

where $\mathcal{C}_*(N_i, N_j)$ is the complex of relative chains as in (4). This short exact sequence induces a long exact homology sequence

$$\cdots \to H_n(N_1, N_0) \to H_n(N_2, N_0) \to H_n(N_2, N_1) \xrightarrow{\partial_n} H_{n-1}(N_1, N_0) \to \cdots.$$

In other words the last long exact sequence is by definition (cf. (4))

$$\cdots \to CH_n(M(I)) \to CH_n(M(IJ)) \to CH_n(M(J)) \xrightarrow{\partial_n} CH_{n-1}(M(I)) \to \cdots. \tag{9}$$

Now we are ready to state the definition of a connection matrix (cf. [Franzosa, 1989, Def. 3.6]), which avoids braids (cf. [Barakat and Robertz, 2009, Def. 2.7]). The definition of a connection matrix relates the algebraically induced long exact sequence (8) and the dynamically induced long exact sequence (9), more precisely:

**Definition 3.5** (Connection matrix)**.** *Let $C = \{CH_*(M(p)) \mid p \in P\}$ be as above and let $\Delta : \bigoplus_{p\in P} CH_*(M(p)) \to \bigoplus_{p\in P} CH_*(M(p))$ be an upper triangular boundary map. $\Delta$ is called a* connection matrix *if for each interval $K \in \mathcal{I}(P,>)$ there exits an isomorphism $\theta(K) : H_*\Delta(K) \to CH_*(M(K))$ such that for all pairs $(I,J) \in \mathcal{I}_2(P,>)$ of adjacent intervals the following diagram*

(10)

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\delta_{n+1}} & H_n\Delta(I) & \longrightarrow & H_n\Delta(IJ) & \longrightarrow & H_n\Delta(J) & \xrightarrow{\delta_n} & H_{n-1}\Delta(I) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \theta(I)} & & \downarrow{\scriptstyle \theta(IJ)} & & \downarrow{\scriptstyle \theta(J)} & & \downarrow{\scriptstyle \theta(I)} & \\
\cdots \xrightarrow{\partial_{n+1}} & CH_n(M(I)) & \longrightarrow & CH_n(M(IJ)) & \longrightarrow & CH_n(M(J)) & \xrightarrow{\partial_n} & CH_{n-1}(M(I)) & \longrightarrow \cdots
\end{array}$$

*is an isomorphism of long exact sequences, i.e. that additionally all the squares commute.*

**Remark 3.6** ([Barakat and Robertz, 2009, Remark 3.4])**.** *We want to emphasize the importance of first choosing a fixed isomorphism $\theta(K)$ for each interval $K$. This single isomorphism enters in all the commutative diagrams (10). Notably, in* Franzosa*'s definition of connection matrices also a fixed isomorphism $\theta(K)$ for each interval $K$ has to be chosen* a priori *(cf.* [Franzosa, 1988, Def. 1.2] *or* [Franzosa, 1989, Def. 2.4]*).*

Following [Barakat and Robertz, 2009], we show that this braid free definition coincides with Franzosa's definition of connection matrices [Franzosa, 1988, Def. 1.4] or [Franzosa, 1989, Def. 3.6]:
In contrast to the above definition of connection matrices, Franzosa's definition requires the isomorphism of two graded module braids. The first braid is obtained as the homology of a chain complex braid in the setup of the upper triangular boundary map $\Delta$ (cf. [Franzosa, 1989, Prop. 3.4] together with [Franzosa, 1989, Prop. 2.7]). The other is obtained as the homology of the chain complex braid of an index filtration, which in turn generalizes our index triples. Salamon proved in [Salamon, 1985] that index filtrations of Morse decompositions always exit, (see also [Franzosa, 1986, Thm. 3.8], [Franzosa and Mischaikow, 1988], and [Mischaikow, 1995, Thm. 4.2.4]). That an index filtration of a Morse decomposition always induces a chain complex braid was proved in [Franzosa, 1986, Section 4], see also the discussion before [Mischaikow, 1995, Def. 4.3.2].

Clearly, and because of the a priori chosen isomorphisms $\theta(K)$, the isomorphism of the long exact sequences in Definition 3.5 gives rise to the isomorphism of the graded module braids, as required by Franzosa.

**Corollary 3.7.** *The definition of connection matrices following* Franzosa [Franzosa, 1988, Def. 1.4] *is equivalent to Definition 3.5 above.*

Franzosa's existence theorem [Franzosa, 1989, Thm. 3.8] guarantees the existence of at least one connection matrix, provided all $CH_*(M(p))$ are free over the coefficient ring. By taking coefficients in a field, as we do by taking $\mathbb{Z}/2\mathbb{Z}$-coefficients, this is immediate.

**Remark 3.8.** *In practice, the lack of topological data, in particular of the index triples on the dynamical side, prevents us from constructing the maps in (9) explicitly. Therefore in the software package* `conley`*, we can only check a part of the defining properties of connection matrices. More precisely, besides* $\Delta$ *being an upper triangular boundary map* [Maier-Paape et al., 2007, Section 3, (C1,C2)]*, we, so far, only check abstract isomorphisms* $H_n\Delta(K) \cong CH_n(M(K))$ *for each interval* $K \in \mathcal{I}(P,>)$ *as in* [Maier-Paape et al., 2007, Section 3, (C3)]. *Accordingly, we call the matrices computed by* `conley` possible *connection matrices.*

## 4. Uniqueness and Non–Uniqueness of connection matrices

In this section we provide some examples of dynamical systems with few equilibria to demonstrate the syntax and abilities of the `conley` package. In particular, we will see how the amount of information provided to the program is crucial when searching for the unique connection matrix — in case there is a unique one.

### 4.1. Franzosa's example for non–uniqueness of connection matrices.

Here we demonstrate how a bifurcation phenomenon could lead to a non–uniqueness of the connection matrix at the bifurcation point, although the whole dynamical data is given. This example is taken from [Franzosa, 1989]. We discuss a 2-dimensional flow with a parameter $\theta > 0$. The relevant parametrized differential equation is:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \theta y - x\left(x - \frac{1}{3}\right)\left(1 - x\right) . \end{aligned} \tag{11}$$

We assume the reader is familiar with the analysis of (11) given in [Franzosa, 1989]. Regardless of $\theta$ we have three equilibria

$$M(1) = M_\theta(1) = \left\{ \left(\frac{1}{3}, 0\right) \right\}, \quad M(2) = M_\theta(2) = \{(0,0)\}, \quad M(3) = M_\theta(3) = \{(1,0)\}$$

with Conley indices (cf. Prop. 3.1)

$$CH_*(M(1)) = \Sigma^0, \quad CH_*(M(2)) = \Sigma^1 \quad \text{and} \quad CH_*(M(3)) = \Sigma^1 .$$

For $0 < \theta' \ll 1$ we get a flow $\varphi'$ indicated by the left of Figure 2 with only one flow–induced relation $2 >_{\varphi'} 1$, whereas for $\theta'' \gg 1$ the flow changes qualitatively to $\varphi''$ as indicated by the right of Figure 2 with flow–induced relations $2 >_{\varphi''} 1$ and $3 >_{\varphi''} 1$.

The two flows $\varphi'$ and $\varphi''$ are abstracted in Figure 3. In between these (abstract) flows a bifurcation, illustrated by Figure 4, takes place.

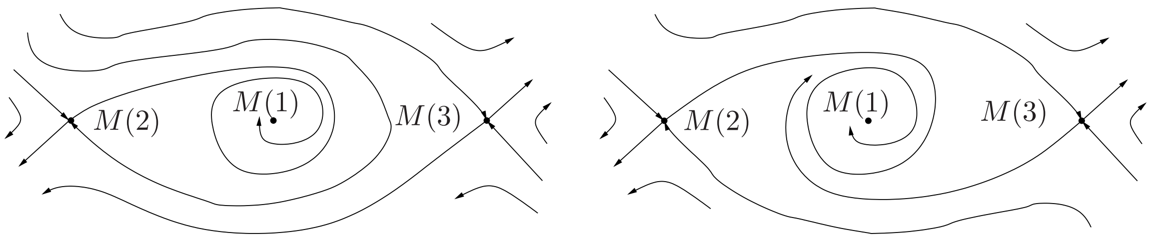



FIGURE 2. Flow for $0 < \theta' \ll 1$ (left) and $\theta'' \gg 1$ (right)

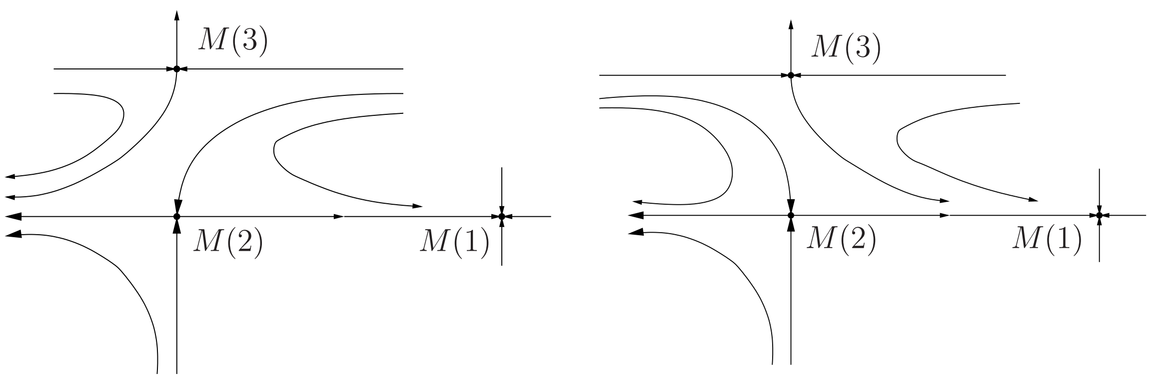



FIGURE 3. Abstract flow for $0 < \theta' \ll 1$ (left) and $\theta'' \gg 1$ (right)

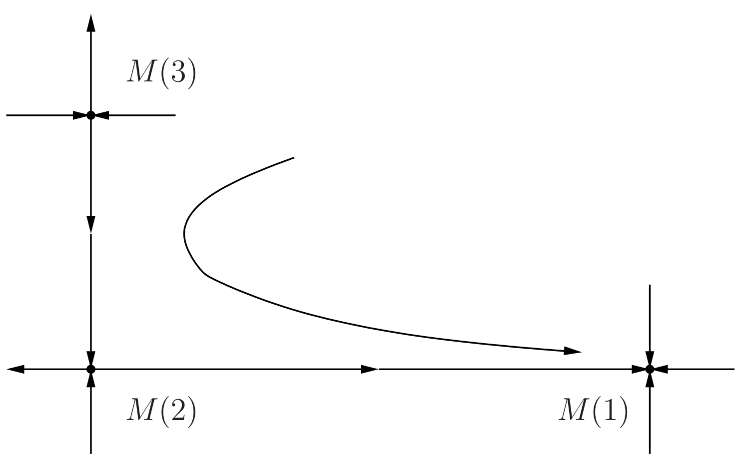



FIGURE 4. Special (abstract) flow at the bifurcation point $\theta_{\text{special}}$

In the following we explain the syntax of the package `conley` by demonstrating a typical Maple session to perform the CONLEY index computations, here relevant to equation (11). A condensed version of this worksheet can be found in the library of examples on the homepage [Barakat et al., 2008] under the name `Franzosa`.

```
> restart;
```

First we load the necessary packages: `conley` is the package containing the CONLEY index algorithms, building upon a general purpose abstract homological algebra library `homalg`. The package `PIR` provides `homalg` with the Maple built-in arithmetics over principal ideal rings.

```
> with(conley):  with(PIR): with(homalg):
> `homalg/default`:=`PIR/homalg`:
```

The principal ideal ring relevant for the following computations is the prime field $\mathbb{Z}/2\mathbb{Z}$.

```
> var:=[2];
```

$$var := [2]$$

```
> Pvar(var);
```

$$[\text{“Z/pZ”}, 2]$$

The three equilibria of the dynamical system (11):

```
> P:=[1,2,3];
```

$$P := [1, 2, 3]$$

We define generating relations of the flow-induced order before the bifurcation $0 < \theta' \ll 1$, i.e. $2 >_{\varphi'} 1$,

```
> rel[0]:=[[2,1]];
```

$$rel_0 := [[2, 1]]$$

at the bifurcation point $\theta_{\text{special}}$,

```
> rel[special]:=[[3,2],[2,1]];
```

$$rel_{special} := [[3, 2], [2, 1]]$$

and after the bifurcation $1 \ll \theta'' < \infty$, i.e. $2 >_{\varphi''} 1$ and $3 >_{\varphi''} 1$:

```
> rel[infinity]:=[[2,1],[3,1]];
```

$$rel_\infty := [[2, 1], [3, 1]]$$

With the below procedure `GeneratePartialOrder` we obtain the order generated by the above relations:

```
> ord[0]:=GeneratePartialOrder(P,rel[0]);
```

$$ord_0 := [[2, 1]]$$

```
> ord[special]:=GeneratePartialOrder(P,rel[special]);
```

$$ord_{special} := [[2, 1], [3, 1], [3, 2]]$$

```
> ord[infinity]:=GeneratePartialOrder(P,rel[infinity]);
```

$$ord_\infty := [[2, 1], [3, 1]]$$

The procedure `GenerateIntervals` returns the set of intervals in $P$ with respect to the given order:

```
> I1[0]:=GenerateIntervals(P,rel[0]);
```

$$I1_0 := [[], [2], [3], [1], [1, 2], [1, 3], [2, 3], [1, 2, 3]]$$

```
> I1[special]:=GenerateIntervals(P,rel[special]);
```

$$I1_{special} := [[], [2], [3], [1], [1, 2], [2, 3], [1, 2, 3]]$$

```
> I1[infinity]:=GenerateIntervals(P,rel[infinity]);
```

$$I1_{\infty} := [[], [2], [3], [1], [1, 2], [1, 3], [2, 3], [1, 2, 3]]$$

In case the CONLEY index of $M(I)$ is known, it may be used for computations. We start with the one-point intervals, i.e. the equilibria. The CONLEY indices of the three equilibria are common for all three flows. $M(1)$ is an attractor with MORSE index 0. $M(2)$ and $M(3)$ are saddle points with MORSE index 1:

```
> CHp:=[0,1,1];
```

$$CHp := [0, 1, 1]$$

Now we provide the specifics of each of the three flows. The lists $CH_i$ are supposed to contain CONLEY indices of specific isolated invariant sets $M(I)$ of the flows ($i = 0, \text{special}, \infty$).

In case $0 < \theta' \ll 1$ we add to the CONLEY indices of the equilibria only the CONLEY index of the (maximal) isolated invariant set $M(P)$, obtaining $CH_0$. `P = 1` means that the whole isolated invariant set has homology CONLEY index $\Sigma^1$:

```
> CH[0]:=[op(zip((x,y)->x=y,P,CHp)),P=1];
```

$$CH_0 := [1 = 0, 2 = 1, 3 = 1, [1, 2, 3] = 1]$$

For $\theta_{\text{special}}$ we add, besides the CONLEY index of $M(P)$, the CONLEY indices of the isolated invariant sets of the intervals $\{1,2\}$ and $\{2,3\}$, obtaining $CH_{\text{special}}$. `[1,2]=[]` means that $M(\{1,2\})$ has the trivial CONLEY index. `[2,3]=[0,2]` means that the homology CONLEY index of $M(\{2,3\})$ is $(0, (\mathbb{Z}/2\mathbb{Z})^2, 0, \ldots)$:

```
> CH[special]:=[op(zip((x,y)->x=y,P,CHp)),[1,2]=[],[2,3]=[0,2],P=1];
```

$$CH_{special} := [1 = 0, 2 = 1, 3 = 1, [1, 2] = [], [2, 3] = [0, 2], [1, 2, 3] = 1]$$

For $1 \ll \theta'' < \infty$ we add besides the CONLEY index of the isolated invariant set $M(P)$, the CONLEY indices of the isolated invariant sets $M(\{1,2\})$ and $M(\{1,3\})$, obtaining $CH_\infty$:

```
> CH[infinity]:=[op(zip((x,y)->x=y,P,CHp)),[1,2]=[],[1,3]=[],P=1];
```

$$CH_{\infty} := [1 = 0, 2 = 1, 3 = 1, [1, 2] = [], [1, 3] = [], [1, 2, 3] = 1]$$

The procedure `ConnectionMatrices` now provides a list of possible connection matrices (cf. [Franzosa, 1989, Mischaikow and Mrozek, 2002]) for the given ordered set and CONLEY index data. The procedure issues an error if any of the sets is not an interval with respect to the ordering. For a given flow and a given MORSE decomposition with admissible ordering of the maximal invariant set each connection matrix will be found. However, the program may return other matrices, in particular, if the CONLEY index data is incomplete. Of course, in case `ConnectionMatrices` returns a single matrix, this is, due to FRANZOSA's existence result [Franzosa, 1989], a (or actually the unique) connection matrix.

For $\theta'$ one obtains a unique connection matrix, although no CONLEY index information about the intervals $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ is used:

```
> Con[0]:=ConnectionMatrices(P,rel[0],CH[0],var,"Hyp"):
> Involution,Con[0],var);
```

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

The option "Hyp" or "Hyperbolic" can only be used if all elementary MORSE sets are hyperbolic equilibria. Since **homalg** uses the row convention by default, we transpose the connection matrices using `Involution`. The output then confirms with the usual notation used in the literature.

As expected, the only possible transition matrix is the identity matrix:

```
> TransitionMatrices(P,rel[0],CHp,var,"Hyp"):
> map(Involution,%,var);
```

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

The procedure `TransitionMatrices` expects as its input only the indices of the equilibria.

Although we use CONLEY index information of all possible intervals, the connection matrix for $\theta_{\text{special}}$ is nevertheless non–unique (this was one of the main achievements of [Franzosa, 1989]):

```
> Con[special]:=ConnectionMatrices(P,rel[special],CH[special],var,
> "Hyp"):
> map(Involution,Con[special],var);
```

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

The group of transition matrices is non-trivial. In contrast to the procedure `TransitionMatrices` the below procedure returns only the generators of the transition matrix group:

```
> Ts:=TransitionMatricesGenerators(P,rel[special],CHp,var,"Hyp"):
> map(Involution,Ts,var);
```

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

The transition matrix group acts transitively on the set of connection matrices by conjugation. The output is a list of orbits (again lists) of connection matrices:

```
> Orb:=Orbits(Ts,Con[special],var,"Conjugation"):
> map(a->map(Involution,a,var),Orb);
```

$$\left[\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]\right]$$

The following is a list of the cardinalities of the different orbits (here only one):

```
> map(nops,Orb);
```

$$[2]$$

For $\theta''$ one obtains a unique connection matrix, although no CONLEY index information about the interval $\{2,3\}$ is used (Remark: the CONLEY index information of the intervals $\{1,2\}$ and $\{1,3\}$ is required for uniqueness):

```
> Con[infinity]:=ConnectionMatrices(P,rel[infinity],CH[infinity],var,
> "Hyp"):
> map(Involution,Con[infinity],var);
```

$$\left[\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

As expected, the only possible transition matrix is the identity matrix:

```
> TransitionMatrices(P,rel[infinity],CHp,var,"Hyp");
```

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

Still for $\theta''$, but with CONLEY index information about a different set of intervals, one obtains non–unique connection matrices:

```
> CH2[infinity] := [1 = 0, 2 = 1, 3 = 1, [1, 2] = [], [2, 3] = [0,2],
> [1, 2, 3] = 1];
```

$$CH2_\infty := [1 = 0, 2 = 1, 3 = 1, [1, 2] = [], [2, 3] = [0, 2], [1, 2, 3] = 1]$$

```
> Con[infinity]:=ConnectionMatrices(P,rel[infinity],CH2[infinity],var,
> "Hyp"):
> map(Involution,Con[infinity],var);
```

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

Note, the only possible transition matrix in this situation remains the identity matrix, since the transition matrix only depends on the index information of the one point intervals.

4.2. FRANZOSA**'s transition matrix example.** Next we look at a gradient flow $\varphi$ serving as a transition system connecting the two systems (11) at $\theta = \theta'$ and $\theta = \theta''$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \theta y - x\left(x - \frac{1}{3}\right)(1 - x) \\ \dot{\theta} &= \varepsilon(\theta' - \theta)(\theta'' - \theta), \end{aligned} \tag{12}$$

with $0 < \theta' \ll 1$ and $1 \ll \theta'' < \infty$ as before and small $\varepsilon > 0$ fixed. This is also studied in [Franzosa, 1989, Example 6.2]. The additional equation for $\theta = \theta(t)$ is decoupled from the others. $\{\theta = \theta'\}$ is invariant and attracting, while $\{\theta = \theta''\}$ is invariant and repelling (cf. Figure 5). A sketch of the combined flow is given in Figure 6.

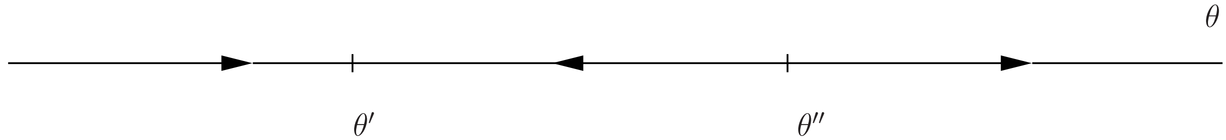



FIGURE 5. The flow in the $\theta$-component

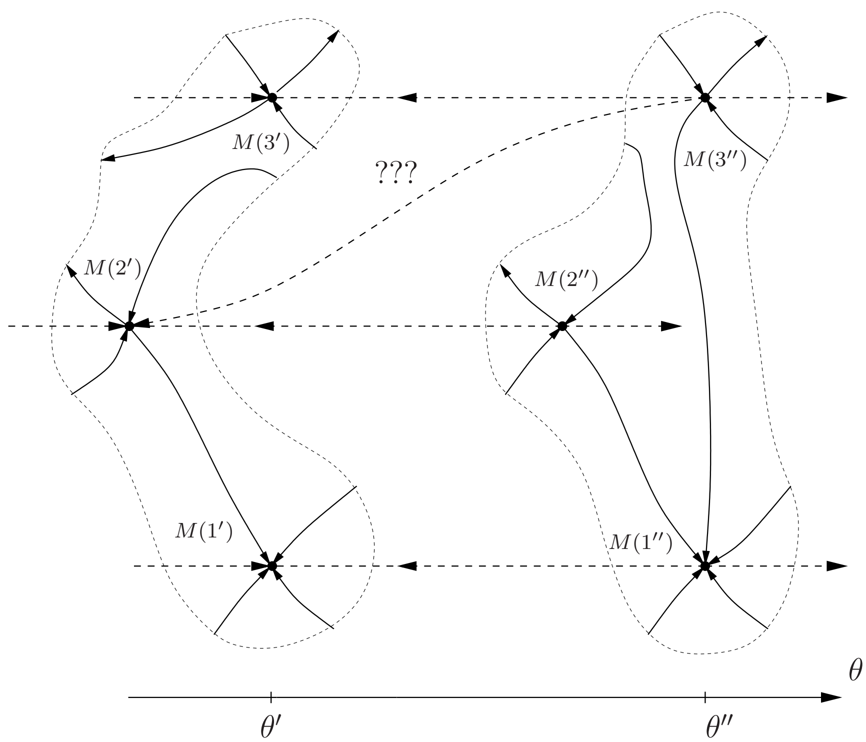



FIGURE 6. The flow $\varphi$ of the extended system

In contrast to Subsection 4.1 we refer for coding details to the worksheet `Franzosa_all` on the homepage [Barakat et al., 2008].

Let $1', 2', 3'$ denote the equilibria for $\theta = \theta'$ and $1'', 2'', 3''$ denote those for $\theta = \theta''$. $1', 2', 3'$ retain their CONLEY indices, while the CONLEY indices of $1'', 2'', 3''$ are raised by one, i.e.

$$\begin{array}{lllllllll} CH_*(M(1')) &=& \Sigma^0, & CH_*(M(2')) &=& \Sigma^1, & \text{and} & CH_*(M(3')) &=& \Sigma^1; \\ CH_*(M(1'')) &=& \Sigma^1, & CH_*(M(2'')) &=& \Sigma^2, & \text{and} & CH_*(M(3'')) &=& \Sigma^2. \end{array} \tag{13}$$

In this worksheet we perform computations with different orders and different CONLEY index data.

**Variant 1:** The flow-induced order $>_\varphi$ (in short $>$) at least contains all connections known for the $\theta = \theta'$ and the $\theta = \theta''$ system, together with the connections between the different copies of the equilibria in the two $\theta$-systems, i.e. we have at least the relations

$$2' > 1', \quad 2'' > 1'', \quad 3'' > 1'', \quad 1'' > 1', \quad 2'' > 2', \quad 3'' > 3'. \tag{14}$$

To the CONLEY indices of the equilibria we only add the CONLEY index of the whole invariant set $M(P)$, which is trivial, i.e. besides (13)

$$CH_*(M(P)) = 0. \tag{15}$$

Apparently we get non–unique connection matrices

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

where each 0 is replaced by a dot. Maybe we didn't provide enough CONLEY index data?

**Variant 2:** We retain the proposed order above, but add CONLEY index information about the "transition"-interval $\{1', 1''\}$ and the $\theta''$-interval $\{1'', 3''\}$. By construction, there is a heteroclinic connection from $i'' \to i'$ with CONLEY index $CH_*(M(\{i', i''\})) = 0$, i.e. the CONLEY data we use additionally to (13) and (15) is

$$CH_*(M(\{1', 1''\})) = CH_*(M(\{1'', 3''\})) = 0. \tag{16}$$

Explicit computations in the worksheet show that there are no connection matrices satisfying the above requirements. But due to FRANZOSA's existence result [Franzosa, 1989, Thm. 3.8] at least one connection matrix is always guaranteed. This inconsistency tells us that our proposed order is not admissible, i.e. that it does not contain the flow-induced order.

The strategy now is to enlarge the order, as a subset of $P \times P$, to avoid inconsistency. The following four possibilities $2'' > 3'$, $3'' > 2'$, $1'' > 2'$, $1'' > 3'$ are in question. The last two can be ruled out immediately, because adding $1'' > 2'$ implies that $\{2', 2''\}$ (and $\{1', 1''\}$) is not anymore an interval, and adding $1'' > 3'$ implies that $\{3', 3''\}$ is no longer an interval. However, the sets $\{i', i''\}$ are always intervals by construction.

**Variant 3:** We add $2'' > 3'$ to the generating relations and retain the enriched CONLEY index information above (13), (15), and (16). Again we run into an inconsistency (no connection matrix matches the data), which tells us that our proposed order is again not admissible, i.e. that it does not contain the flow induced order.

**Variant 4:** The only possible enlargement left is the relation $3'' > 2'$. This indeed proves that the flow-induced order $>_\varphi$ is generated by the relations (14) together with $3'' > 2'$.

Our computations yield the unique connection matrix

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Thus, the above inconsistency is resolved. Since furthermore $(\{2'\}, \{3''\}) \in \mathcal{I}_2(P, >_\varphi)$ is a pair of adjacent intervals and $\Delta(2', 3'') \neq 0$ (cf. [Maier-Paape et al., 2007, Section 3,(C4)]), the existence of a connecting orbit $3'' \to 2'$ is proved. This connecting orbit was already found in FRANZOSA's original article (see [Franzosa, 1989, Example 6.2]). The line of arguments provided above is nevertheless new.

## 5. The Cahn-Hilliard Equation Revisited

Here we pursue the discussion started in [Maier-Paape et al., 2007] of the CAHN-HILLIARD equation

$$\begin{aligned} u_t &= -\Delta(\varepsilon^2 \Delta u + f(u)) \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{17}$$

where $\Omega = (0,1)^2$ is the unit square, $f(u) = u - u^3$, and $\varepsilon > 0$ is a parameter. The CAHN-HILLIARD equation is a gradient flow with respect to the VAN DER WAALS free energy function

$$E_\varepsilon[u] := \int_\Omega \left( \frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x, \tag{18}$$

where $F(u) = (u^2 - 1)^2/4$. The equilibria of the parametric CAHN-HILLIARD equation are described in [Maier-Paape et al., 2007, Section 4.2, Prop. 4.5].

The set of all functions on $\Omega$ satisfying the integral condition $\int_\Omega u \mathrm{d}x = m$, for a fixed mass $m \in \mathbb{R}$, is invariant under the flow. The constant function $u \equiv m$ is an equilibrium of the CAHN-HILLIARD equation, which we simply denote by $m$. We introduce the parameter $\lambda = 1/\varepsilon^2$.

### 5.1. Numerically Motived Hypotheses.

Figure 7 is taken from [Maier-Paape et al., 2007, Figure 23] and shows the bifurcation diagram of the equilibria for $m = 0$ and $0 < \lambda \leq \lambda^*_{4,0}$. More precisely the numerically motivated hypotheses corresponding to this figure have been summed up as (H1)'-(H6)' in [Maier-Paape et al., 2007, Section 6.1].

In Figure 8 typical nodal lines of the equilibria occurring in the diagram in Figure 7 are sketched.

Figure 9 shows the bifurcation diagram for $m < 0$ small in the range $0 < \lambda \leq \lambda^*_{4,m}$. Here we assume numerically motivated hypotheses similar to (H1)' - (H6)' from [Maier-Paape et al., 2007,

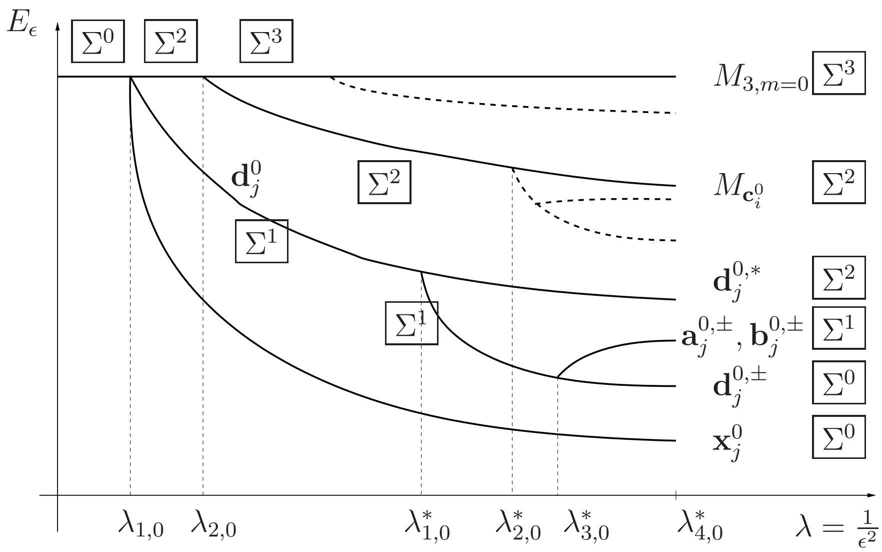



FIGURE 7. Bifurcation diagram and energy values for $m = 0$

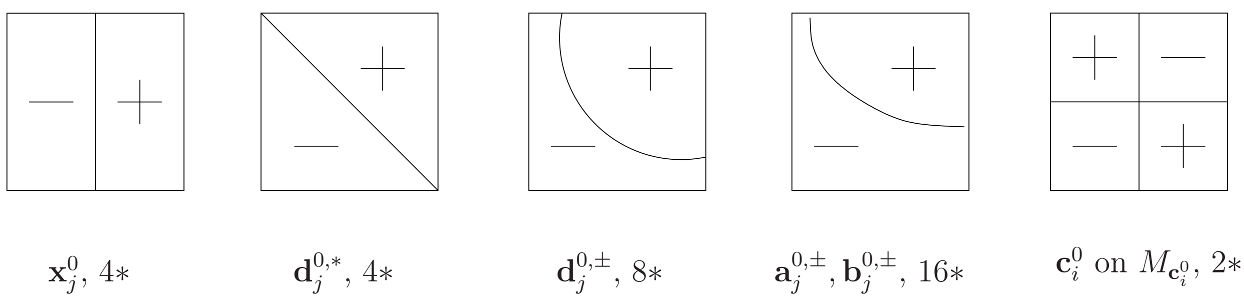



FIGURE 8. Pattern associated to the main branches of $m = 0$ and their multiplicity.

Subsection 6.1] and in the obvious way for $m < 0$ small adapted according to Figure 9. For details cf. [Maier-Paape et al., 2008].

The nodal lines of the equilibria branches occurring in the energy diagram are sketched in Figure 10 and 11.

5.2. **The Range $\lambda_{1,m} < \lambda < \lambda_{2,m}$.** For $0 \leq m^2 < \frac{3}{35}$ there exist nine equilibria

$$\{x_0^m, x_1^m, x_2^m, x_3^m, d_0^m, d_1^m, d_2^m, d_3^m, m\}$$

of the equation (17) subject to $\int_\Omega u dx = m$ for $\lambda_{1,m} < \lambda < \lambda_{2,m}$ as described in [Maier-Paape et al., 2007, Subsections 2.2 and 2.3]. See Figure 9. We refer to the worksheet `Cahn-Hilliard` on the homepage [Barakat et al., 2008] for the details of these computations and only indicate the relevant input:

```
> P:=[ x0,x1,x2,x3, d0,d1,d2,d3, m];
```

The CONLEY indices of the nine equilibria (the $x_i^m$'s are stable, and the $d_i^m$'s resp. $m$ are saddle points with one resp. two unstable directions; cf. Prop. 3.1):

```
> CHp:=[ 0,0,0,0, 1,1,1,1, 2];
```

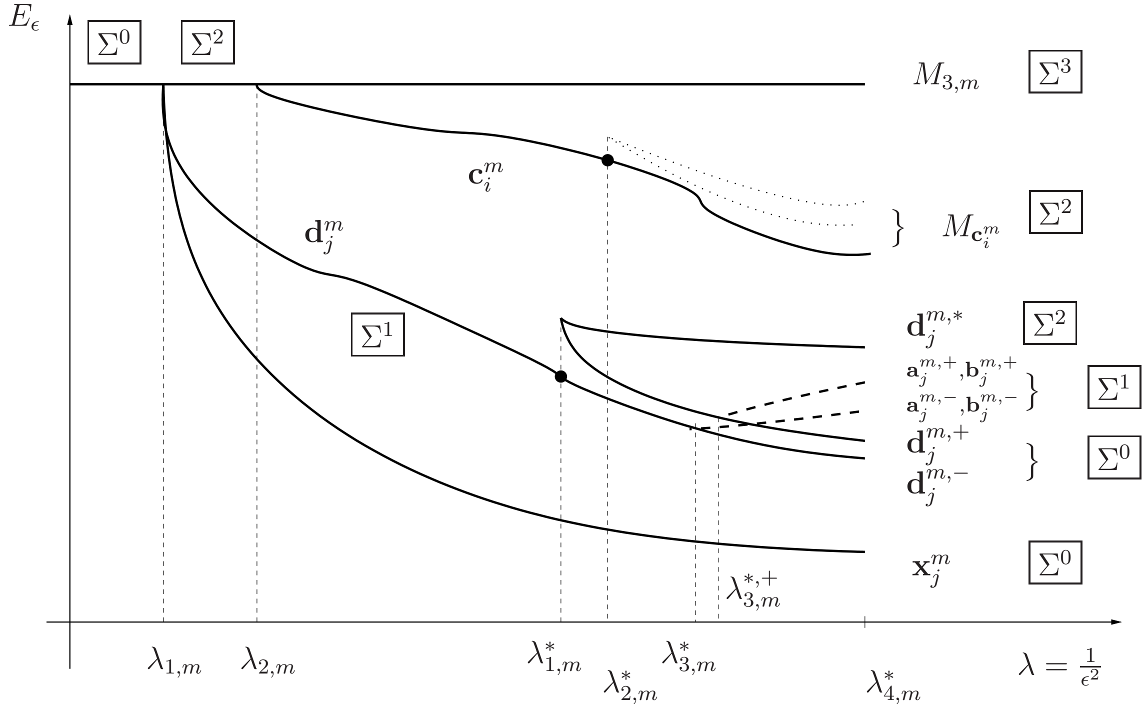



FIGURE 9. Partial bifurcation diagram and energy values for small $m < 0$

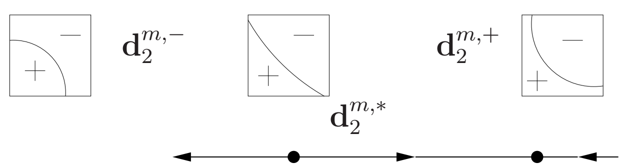



FIGURE 10. $\mathbf{d}_2^{m,-}$ and the saddle node branch ($\mathbf{d}_2^{m,*}$ and $\mathbf{d}_2^{m,+}$) for small $m < 0$

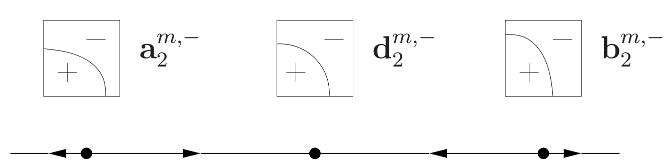
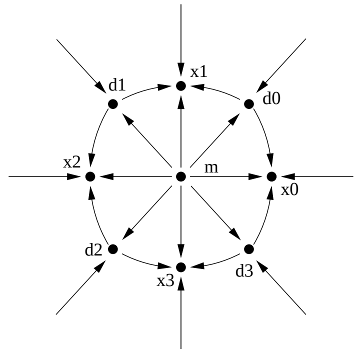



FIGURE 11. $\mathbf{d}_2^{m,-}$ bifurcates to $\mathbf{a}_2^{m,-}$ and $\mathbf{b}_2^{m,-}$ for small $m < 0$

The symmetry group of the dynamics




is the dihedral group $D_8 := \langle (x_0^m, x_1^m, x_2^m, x_3^m)(d_0^m, d_1^m, d_2^m, d_3^m), (x_1^m, x_3^m)(d_0^m, d_3^m)(d_1^m, d_2^m) \rangle$:

```
> D8:=[ [[x0,x1,x2,x3],[d0,d1,d2,d3]], [[x1,x3],[d0,d3],[d1,d2]] ];
```

Now we interpret the MORSE indices as abstract energy levels giving $m > d_i^m > x_j^m$. Because of the $D_8$-symmetry the $x_i^m$'s resp. $d_i^m$'s are on the same energy level (18) and thus the $x_i^m$'s resp. $d_i^m$'s are not related among each other in the energy–induced, and hence also in the flow–induced order. The energy–induced order:

```
> rel:=DefinePartialOrderByPotential(P,CHp,P);
```

$rel := [[d0, x0], [d1, x0], [d2, x0], [d3, x0], [m, x0], [d0, x1], [d1, x1], [d2, x1], [d3, x1], [m, x1], [d0, x2], [d1, x2], [d2, x2], [d3, x2], [m, x2], [d0, x3], [d1, x3], [d2, x3], [d3, x3], [m, x3], [m, d0], [m, d1], [m, d2], [m, d3]]$

To the CONLEY indices of the equilibria we only add the CONLEY index of the isolated invariant set $M(P)$, which is a global attractor $\mathcal{A}_\lambda^m$, depending on the mass $m$ and the parameter $\lambda$. By Prop. 3.2 the homology CONLEY index is $CH_*(\mathcal{A}_\lambda^m) = \Sigma^0$:

```
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=0];
```

As in [Barakat and Robertz, 2009, Example 7.3] we compute using `conley` the relevant connection matrices, where now we additionally impose the "extra" dynamical condition that for $\lambda$ greater but close to $\lambda_{1,m}$ there are no connecting orbits between $d_1^m$ and $x_0^m$ (cf. [Maier-Paape et al., 2007, Lemma 2.2]):

```
> Con:=ConnectionMatrices(P,rel,CH,var,"Symmetry"=D8,
> "Extra"=[[d1,x0]=0]);
```

$$\left[\left[\left[\begin{array}{cccc} 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 \end{array}\right], \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

This result coincides with [Maier-Paape et al., 2007, Prop. 4.5]. Contrary to the hyperbolic notation for connection matrices used in Subsection 4.1 we prefer here the so-called *standard notation*. The procedure `ConnectionMatrices` defaults to this notation if no other options are provided. Each connection matrix is now returned as the tuple $\Delta = (\Delta_n)_{n=1,2,\ldots}$, where $\Delta_n : C_n(P) \to C_{n-1}(P)$.

Compared with [Barakat and Robertz, 2009, Example 7.3], the additional dynamical condition used here gives a *unique* connection matrix (for $\lambda$ greater but close to $\lambda_{1,m}$). And this remains true for all $\lambda_{1,m} < \lambda < \lambda_{2,m}$ since the transition matrix group is trivial:

```
> Ts:=TransitionMatrices(P,rel,CHp,var);
```

$$\left[\left[\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}\right]\right]$$

Again, contrary to the hyperbolic notation for transition matrices used in Subsection 4.1 we use the standard notation. The procedure `TransitionMatrices` also defaults to this notation if no other options are provided. Each transition matrix is returned as the tuple $T = (T_n)_{n=0,1,\ldots}$, where $T_n : C_n(P) \to C_n(P)$.

5.3. **The Range** $\lambda_{2,m} < \lambda < \lambda^*_{1,m}$. In this range[1] two new equilibria $c_1^m$ and $c_2^m$ with Morse index 2 bifurcate from the constant solution (see Figure 9):

```
> P:=[x0,x1,x2,x3, d0,d1,d2,d3, c0,c1, m];
```

The Morse index of the constant solution is now raised to 3, where $m$ may stand for a non-trivial isolated invariant set $M_3 = M_{3,m}(\lambda)$ with Conley index $\Sigma^3$ that contains the trivial solution $u \equiv m$ (cf. [Maier-Paape et al., 2007, Subsection 2.4,(H5)]):

```
> CHp := [0,0,0,0, 1,1,1,1, 2,2, 3];
```

The partial order (here we abuse the Morse indices as abstract energy levels to define an energy induced order):

```
> rel:=DefinePartialOrderByPotential(P,CHp,P);
```

For general $m$ the symmetry group is $D_8$ as before the bifurcation, now also acting on the two new equilibria $c_1^m, c_2^m$:

```
> D8 := [[[x0,x1,x2,x3],[d0,d1,d2,d3],[c0,c1]],
> [[x1,x3],[d0,d3],[d1,d2],[c0,c1]]];
```

For $m = 0$ the symmetry group is larger; multiplying with $-1$ is an additional symmetry:

```
> D8C2:= [ op(D8), [[x0,x2],[x1,x3],[d0,d2],[d1,d3],[c0,c1]] ];
```

Even for the $m \neq 0$, i.e. only using the $D_8$-symmetry, we obtain a unique connection matrix as in [Maier-Paape et al., 2007, Prop. 4.6]:

```
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=0];
> Con0:=ConnectionMatrices(P,rel,CH,var,"Symmetry"=D8,
> "Extra"=[[d1,x0]=0]);
```

$$\left[\left[\begin{bmatrix} 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

[1] We refer to the worksheet `CH[2m,1m_star]` on the homepage [Barakat et al., 2008] for the details of these computations

Since this matrix is unique, computing with the bigger symmetry group cannot bring any further restriction.

```
> Ts:=TransitionMatricesGenerators(P,rel,CHp,var,"Symmetry"=D8);
```

$$Ts := \left[\left[\left[\begin{smallmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&\cdot\\\cdot&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\end{smallmatrix}\right]\right]\right]$$

As the transition matrix group is trivial, the above connection matrix remains unique on the whole interval where this Morse decomposition exists.

5.4. **The Range $\lambda^*_{1,m} < \lambda < \lambda^*_{2,m}$.** Here we compute[2] the matrices found in [Maier-Paape et al., 2007, Section 5]. We discuss $\lambda$ in the range $\lambda^*_{1,m} < \lambda < \lambda^*_{2,m}$ as described in [Maier-Paape et al., 2007, Subsections 2.2 and 2.3]. For $|m|$ small there exist 18 equilibria[3] and one non-trivial Morse set corresponding to $m$ as above:

$$\{x_0^m, x_1^m, x_2^m, x_3^m, d_0^{m,-}, d_1^{m,-}, d_2^{m,-}, d_3^{m,-}, d_0^{m,+}, d_1^{m,+}, d_2^{m,+}, d_3^{m,+}, d_0^{m,*}, d_1^{m,*}, d_2^{m,*}, d_3^{m,*}, c_0^m, c_1^m, m\}$$

of the Cahn-Hilliard equation:

```
> P:=[x0,x1,x2,x3, dm0,dm1,dm2,dm3, dp0,dp1,dp2,dp3,
> ds0,ds1,ds2,ds3, c0,c1, m];
```

The energy induced order for $m = 0$. Note that although the $c_i^m$'s and the $d_j^{m,*}$ have the same Morse index 2, the energy of the first ones is higher than the energy of the others (see Figure 7):

```
> vm_0:=[0,0,0,0, 1,1,1,1, 1,1,1,1,
> 2,2,2,2, 2.5,2.5, 3];
```

The abstract energy level 2.5 of the $c_i^m$'s is used to define the partial order (our convention is to choose the abstract energy levels in such a way, that its integer part equals its Morse index):

```
> relm_0:=DefinePartialOrderByPotential(P,vm_0,P);
```

The energy induced order for $m < 0$ (for $m < 0$ all the $d_i^{m,-}$ have lower energy than all the $d_i^{m,+}$, but both have Morse index 1, see Figure 9):

```
> vm_neg:=[0,0,0,0, 1,1,1,1, 1.5,1.5,1.5,1.5,
> 2,2,2,2, 2.5,2.5, 3];
```

Note that the energy induced order for $m < 0$ is admissible for $m = 0$, too.

```
> relm_neg:=DefinePartialOrderByPotential(P,vm_neg,P);
```

The direct factor $C_2$ acts by $-1$ on the set of functions. It only exists in the $m = 0$ case:

[2] We refer to the worksheet `CH_On_Square` on the homepage [Barakat et al., 2008] for the details of these computations

[3] If one doesn't resolve the bifurcation at $c_i^m$, similar results hold true in the range $\lambda^*_{2,m} < \lambda < \lambda^*_{3,m}$.

```
> D8C2:=
> [
> [[x0,x1,x2,x3],[dm0,dm1,dm2,dm3],[dp0,dp1,dp2,dp3],
> [ds0,ds1,ds2,ds3],[c0,c1]],
> [[x1,x3],[dm0,dm3],[dm1,dm2],[dp0,dp3],[dp1,dp2],
> [ds0,ds3],[ds1,ds2],[c0,c1]],
> [[x0,x2],[x1,x3],[dm0,dp2],[dm1,dp3],[dm2,dp0],[dm3,dp1],
> [ds0,ds2],[ds1,ds3],[c0,c1]]
> ];
```

In the $m \neq 0$ case the symmetry is broken from $D_8 \times C_2$ down to $D_8$:

```
> D8:=D8C2[1..2];
```

The CONLEY indices of the respective equilibria (for all $|m|$ small):

```
> CHp:=[0,0,0,0, 1,1,1,1, 1,1,1,1,
> 2,2,2,2, 2,2, 3];
```

We only add the condition saying $M(P)$ is a global attractor:

```
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=0];
```

5.4.1. *The case $m = 0$.* Here we discuss the $m = 0$ case. In the following computation the condition $a = 1$ (`[dm0,x0]=1`) is algebraically equivalent to $b = 0$ (`[dm1,x0]=0`). The condition $a = 1$ follows from a dynamical argument near the bifurcation point $\lambda = \lambda^*_{1,0}$, cf. [Maier-Paape et al., 2007, Proof of Thm. 5.3] (see [Maier-Paape et al., 2007, (34) and (35)] for the definition of the constants $a, b, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\alpha}, s$). The other two dynamical conditions $\widetilde{\beta} = \widetilde{\gamma} = 0$ (`[ds1,dp0]=[ds2,dp0]=0`) are dictated by [Maier-Paape et al., 2007, Prop. 5.2], where the role of the tilde and non-tilde variables is exchanged due to [Maier-Paape et al., 2007, Lemma. 5.1] and $m < 0$:

```
> Con0:=ConnectionMatrices(P,vm_0,CH,var,"Symmetry"=D8C2,
> "Extra"=[[dm1,x0]=0,[ds1,dp0]=0,[ds2,dp0]=0]);
```

$$
\left[\left[\begin{bmatrix} 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right],\right.
$$

$$
\left.\left[\begin{bmatrix} 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]
$$

The two connection matrices obtained agree with [Maier-Paape et al., 2007, Thm. 5.3]. Note that our computation shows that one doesn't need to impose the affine conditions $\widetilde{\alpha} = s = 1$ in [Maier-Paape et al., 2007, Prop. 5.2]!

The non-uniqueness of the output of `ConnectionMatrices` can have dynamical reasons as in Subsection 4.1, or can be due to the incompleteness of the provided dynamical data as in Remark 3.8. Hence, the output of `ConnectionMatrices` might very well be a strict super set of the set of connection matrices. Here and in the following we therefore call the output of `ConnectionMatrices` the *set of possible connection matrices*.

Based on non-rigorous numerical calculations of the very specific heteroclinic orbit between $c_0^0$ and $d_0^{0,+}$, we are led to the assumption that only the second matrix in `Con0` is a, and hence the unique connection matrix [Maier-Paape et al., 2007, Cor. 5.4].

For $m = 0$ the transition matrix group is the cyclic group $C_2$ with the generator:

```
> Ts0:=TransitionMatricesGenerators(P,vm_0,CHp,var,"Symmetry"=D8C2);
```

$$\left[\left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&1&1\\\cdot&1&\cdot&\cdot&1&1\\\cdot&\cdot&1&\cdot&1&1\\\cdot&\cdot&\cdot&1&1&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right]\right]$$

The transition matrix group interchanges the two connection matrices as supposed by [Maier-Paape et al., 2007, Thm. 5.3]:

```
> Orb0:=Orbits(Ts0,Con0,var,"conjugation"):
> map(nops,Orb0);
```

$$[2]$$

The last line means that acting with the transition matrix group on the set of connection matrices we get a single orbit of length 2.

5.4.2. *The case $m < 0$ small.* Now we compute the transition matrices for $m \leq 0$. Here we use the energy induced order from $m < 0$, which is, as noted above, admissible for $m = 0$, too.

```
> Ts:=TransitionMatricesGenerators(P,vm_neg,CHp,var,"Symmetry"=D8);
```

$$\Bigg[\left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right], \left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right],$$

$$\left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right], \left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&1\\\cdot&\cdot&1&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right],$$

$$\left[\begin{bmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&1&\cdot\\\cdot&\cdot&1&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&1&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{bmatrix}, \begin{bmatrix}1\end{bmatrix}\right]\Bigg]$$

The transition matrix group is thus generated by 5 elements. It is an abelian group (the orbits of the generators under conjugation are one point sets) of order 32. Since each generator has order 2, we conclude that the transition matrix group is isomorphic to the elementary abelian group $C_2^5$.

According to [Franzosa and Mischaikow, 1998] or [Maier-Paape et al., 2007, Prop. 3.5] the set of connection matrices in the range $m \leq 0$ must be generated by conjugating the two for $m = 0$ with the transition matrix group given above. We get 32 such matrices. These matrices are indeed possible connection matrices for $m = 0$, *not* taking into account the full symmetry group $D_8 \times C_2$. Similarly they are connection matrices for $m < 0$, and as we see in the worksheet, they are invariant under the symmetry group $D_8$ for $m < 0$.

First we compute the connection matrices for $m < 0$, only using the three homogeneous dynamical restrictions as above. All three are due to [Maier-Paape et al., 2007, Prop. 5.2].

```
> ConH:=ConnectionMatrices(P,vm_neg,CH,var,"Symmetry"=D8,
> "Extra"=[[dm1,x0]=0,[ds1,dp0]=0,[ds2,dp0]=0]);
```

Since we didn't plug in all 6 dynamical conditions of [Maier-Paape et al., 2007, Prop. 5.2], but only three homogeneous among them, we obtain twice as much as the 32 possible connection matrices from above. Surprisingly, only *one* of the extra affine condition suffices to obtain the correct number 32. If we plug in all 6 conditions right away, then the computation becomes considerably faster, and we again end up with our 32 possible connection matrices.

5.4.3. *Further reduction using dynamical arguments (still $m < 0$ small).* In the hyperbolic context, one can use the following ad hoc argument for index 1 equilibria reducing the number of possible connection matrices even further (this is derived in [Maier-Paape et al., 2007, Section 5.3] on a particular example, but applies to more general situations as described below). We will show in 5.4.4 how one can reduce this to a conventional algebraic Conley index argument.

This ad hoc argument starts with the following observation: since there is only one unstable direction, one has exactly two connecting orbits starting from an index 1 equilibrium.

In this example we consider the four index 1 equilibria $d_i^{m,+}$ for $m < 0$. Shortly after the bifurcation $(m = 0) \to (m < 0)$ one can assume that the $d_i^{m,+}$'s and the $x_j^m$'s are still pairwise adjacent in the flow induced order (although not anymore in the energy induced order), since this was the case for $m = 0$ (cf. [Mischaikow and Mrozek, 2002, Prop. 1.1]).

Consider one $q := d_i^{m,+}$ for a fixed $i$ and the set of $x_j^m$'s ($j = 0, \ldots, k = 3$), which are all index 0 equilibria, where possible connections have to considered. For $k > 2$ there are in general four cases to distinguish. Note that the below argument applies to general index $\Sigma^1$ to index $\Sigma^0$ connections:

**Case I:** There are three or more 1-entries in the connection matrix between $q$ and the $x_j^m$'s. This gives three or more heteroclinic connections starting at $q$, contradicting the above argument (cf. [McCord, 1988]).

**Case II:** There are precisely two 1-entries in the connection matrix between $q$ and the $x_j^m$'s. This is possible and would mean that there are two heteroclinic orbits that start at $q$ and go to different $x_j^m$'s.

**Case III:** There is a single 1-entry in the connection matrix between $q$ and the $x_j^m$'s. This would yield an uneven number of heteroclinic orbits starting at $q$, which is again impossible.

**Case IV:** There is no 1-entry in the connection matrix between $q$ and the $x_j^m$'s. This is in principle possible, but would mean that the two heteroclinic orbits starting at $q$ go to the same $x_j^m$ (characteristic 2). This can be ruled out in case the isotropy subgroup of $q$ doesn't fix any of the $x_j^m$'s, since then the two heteroclinic orbits from above get mapped by the symmetry operation to at least two other heteroclinic orbits, which results in four or more heteroclinic orbits starting at $q$. Again a contradiction.

This helps us to reduce the number of possible connection matrices further. Case I rules out connection matrices like the following one:

```
> map(Involution,Con[3],var);
```

$$\left[\begin{bmatrix} 1 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}\right]$$

There are 8 such connection matrices (out of a total of 32). Now consider for instance:

```
> map(Involution,Con[1],var);
```

$$\left[\begin{bmatrix} 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix}\right]$$

In order to apply Case III from above we have to compute the orbit of connections starting at $d_0^{m,+}$ and see that the isotropy group $\mathrm{Stab}_{D_8}(d_0^{m,+}) \leq D_8$ doesn't fix any of the $x_j^m$'s:

```
> Orbits(D8,[[dp0,x0]]); Orbits(D8,[[dp0,x2]]);
```

$$[[[dp0,\ x0],\ [dp1,\ x1],\ [dp3,\ x0],\ [dp2,\ x2],\ [dp2,\ x3],\ [dp0,\ x1],\ [dp3,\ x3],\ [dp1,\ x2]]]$$

$$[[[dp0,\ x2],\ [dp1,\ x3],\ [dp3,\ x2],\ [dp2,\ x0],\ [dp2,\ x1],\ [dp0,\ x3],\ [dp3,\ x1],\ [dp1,\ x0]]]$$

There are again 8 such connection matrices (out of a total of 32). Hence, we are left with 16 relevant connection matrices. Note that this is still a superset of the 8 connection matrices obtained in [Maier-Paape et al., 2007, Theorem 5.6]. The argument needed to get down to 8 connection matrices is still unavoidable in the alternative algebraic approach below, and we will explain it there.

5.4.4. *Replacing the above dynamical argument by an algebraic argument.* In the following we show how the argument explained above can be replaced by a conventional algebraic argument using the CONLEY index of the interval $J := \{x_0^m, x_1^m, x_2^m, x_3^m, d_0^{m,+}, d_1^{m,+}, d_2^{m,+}, d_3^{m,+}\}$.

For $m = 0$ the set $J$ is an interval and therefore $M(J)$ is an isolated invariant set [Mischaikow and Mrozek, 2002, Prop. 2.12]:

```
> IsInterval(J,P,relm_0);
```

*true*

Using conley we compute that for $m = 0$ the interval $J$ has the index $H_*\Delta(J) = (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0, \ldots)$ for *both* connection matrices in Con0 of Subsection 5.4.1:

```
> HomologyModules(IntervalChainComplex(Con0[1],P,J,CHp,var),var);
> HomologyModules(IntervalChainComplex(Con0[2],P,J,CHp,var),var);
```

Hence we conclude that $CH_*(M(J)) = (\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0, \ldots)$.

According to [Mischaikow and Mrozek, 2002, Prop. 1.1] the isolated invariant set $M(J)$ survives for $|m|$ small and by [Mischaikow and Mrozek, 2002, Theorem 3.10] its index remains unchanged. Note that therefore $J$ survives as an interval in the flow induced order in the passage from $m = 0$ to $m$ negative and small. In the energy induced order, this is no longer the case:

```
> IsInterval(J,P,relm_neg);
```

*false*

But since the energy induced order at $m = 0$ is an admissible ordering for $m < 0$ small, we use it in our computation.

```
> CH_J := [op(CH),J=[1,1]];
```

With this enriched Conley index data we find 16 connection matrices. We will use the affine conditions $a = 1$ ([dm0,x0]=1), $\widetilde{\alpha} = 1$ ([ds0,dp0]=1), and $s = 1$ ([m,c0]=1) from [Maier-Paape et al., 2007, Prop. 5.2]. The input syntax of affine conditions (opposed to linear conditions) is more involved due to the graded structure of $\Delta$. The correct input for the affine condition [dm0,x0]=1 is [dm0,x0]=[ [[1]],[[0]],[[0]] ], i.e. a 1 in the matrix $\Delta_1 : C_1(P) \to C_0(P)$ at the position $(d_0^{m,-}, x_0^m)$.

```
> Con16:=ConnectionMatrices(P,vm_0,CH_J,var,"Symmetry"=D8,
> "Extra"=[[dm1,x0]=0,[ds1,dp0]=0,[ds2,dp0]=0,
> [dm0,x0]=[ [[1]],[[0]],[[0]] ],[ds0,dp0]=[ [[0]],[[1]],[[0]] ],
> [m,c0]=[ [[0]],[[0]],[[1]] ]]):
```

Thus, we have replaced the ad hoc argument in 5.4.3 by a conventional Conley index argument.

In the following we explain in some detail the argument used in [Maier-Paape et al., 2007, Section 5.3] to get further down to 8 connection matrices in the case $m < 0$ small:

First note that the computation of the generators of the transition matrix group in the conley package is done in the following way: Over $\mathbb{Z}/2\mathbb{Z}$ the diagonal entries of the invertible lower triangular matrices must be equal to 1. After subtracting the identity the resulting matrices are completely characterized by being strict lower triangular of degree 0, which are invariant under the symmetry group for $m < 0$. In order to get a generating set of the group, we substitute a *single* 1 at an admissible position $(p, q)$ (i.e. if $p > q$) and add further 1's at (admissible) positions $(p', q')$, iff $(p', q')$ lies in the orbit of $(p, q)$ under the symmetry group. These generators are for the CONLEY index theory *canonical*, since they describe a bifurcation of a *single* switching heteroclinic connection (cf. [Franzosa, 1989]), see 5.4.2, where the canonical generators have been computed.

We exclude those canonical generators, that take connection matrices at $m = 0$ to those connection matrices $\tilde{\Delta}$ violating the CONLEY index condition $H_*\tilde{\Delta}(J) = CH_*(M(J))$ for the isolated invariant set $M(J)$. These are explicitly the first and third generator in `Ts` of Subsection 5.4.2. The remaining three canonical generators generate the elementary abelian group $C_2^3$ of order 8. Applying this transition matrix group to the two connection matrices at $m = 0$ we get the 8 possible connection matrices for $m < 0$ small of [Maier-Paape et al., 2007, Subsection 5.3].

The procedure `TransitionMatricesOfBranching` automatizes all the above steps necessary to compute the three relevant canonical *generators* of the transition matrix group:

```
> obj:=P,vm_0,vm_neg,CHp,CH,CH_J,var,"Symmetry"=D8C2,
> "Symmetry"=D8,"Extra"=[[dm1,x0]=0,[ds1,dp0]=0,[ds2,dp0]=0]:
> G:=TransitionMatricesOfBranching(obj);
```

The extra conditions are only used to calculate `Con0` internally. Applying the above three relevant canonical transition matrix generators to the two connection matrices at $m = 0$ we obtain the 8 possible connection matrices mentioned above:

```
> Orbits(G,Con0,var,"conjugation");
```

The above two steps can finally be summed up in the procedure `ConnectionMatricesOf-Branching` taking the same input as `TransitionMatricesOfBranching`:

```
> ConB:=ConnectionMatricesOfBranching(obj);
```

There are two remarkable issues about $J$:

(1) $J$ is an interval for $m = 0$, which is no longer an interval for $m < 0$ *in the energy induced orders*.
(2) $CH_*(M(J)) = H_*\Delta(J)$ for all possible connection matrices $\Delta$ at $m = 0$.

Note, that we have verified (2) even for the (super) set `Con0` of *possible* connection matrices (cf. Subsection 5.4.1). Therefore in this situation we don't even need to a priori know the CONLEY index $CH_*(M(J))$ because it can automatically be provided by `conley`.

**Definition 5.1** (Energy induced bifurcation interval)**.** *We call such intervals* energy induced bifurcation intervals. *The* relevant *canonical generators of the transition matrix group are those which respect the* CONLEY *index data for all possible bifurcation intervals.*

In this subsection it sufficed to consider only one energy induced bifurcation interval, namely $J$.

5.5. **The Range** $\lambda_{2,m}^* < \lambda < \lambda_{3,m}^*$. Using the numerically motived hypotheses for $m = 0$ and $m < 0$ small stated in Subsection 5.1 above, analogously to corollary [Maier-Paape et al., 2007, Cor. 6.2], all the connection matrices from the previous Subsection remain unchanged, only the branches corresponding to $c_i^m$ get replaced by the isolated invariant set $M_{c_i^m}$.

5.6. **The Range $\lambda^*_{3,m} < \lambda < \lambda^*_{4,m}$.** In [Maier-Paape et al., 2007, Section 6] only the case $m = 0$ was studied. Here we compute the connection matrices and transition matrices also for $m < 0$ small. We discuss $\lambda$ in the range $\lambda^*_{3,m} \le \lambda^{*,+}_{3,m} < \lambda < \lambda^*_{4,m}$ as shown in Figure 9. For $|m|$ small there exist 32 equilibria and 3 non-trivial MORSE sets

$$\{x_0^m, x_1^m, x_2^m, x_3^m, d_0^{m,-}, d_1^{m,-}, d_2^{m,-}, d_3^{m,-}, d_0^{m,+}, d_1^{m,+}, d_2^{m,+}, d_3^{m,+},$$
$$a_0^{m,-}, b_0^{m,-}, a_1^{m,-}, b_1^{m,-}, a_2^{m,-}, b_2^{m,-}, a_3^{m,-}, b_3^{m,-}, a_0^{m,+}, b_0^{m,+}, a_1^{m,+}, b_1^{m,+}, a_2^{m,+}, b_2^{m,+}, a_3^{m,+}, b_3^{m,+},$$
$$d_0^{m,*}, d_1^{m,*}, d_2^{m,*}, d_3^{m,*}, M_{c_0^m}, M_{c_1^m}, M_{3,m}\}$$

of the CAHN-HILLIARD equation. Note that in the range $\lambda^*_{3,m} < \lambda^{*,+}_{3,m}$ the branches $a_i^{m,+}, b_j^{m,+}$ do not yet exist ($m < 0$ small), cf. [Maier-Paape et al., 2008]. The MORSE sets $M_{c_0^m}, M_{c_1^m}, M_{3,m}$ related to c0, c1, m will not be resolved any further, i.e. although there are bifurcations from $c_i^m$ and the the constant solution $m$, we keep the main branches including everything emerging from it as a big MORSE set:

```
> P:=[x0,x1,x2,x3, dm0,dm1,dm2,dm3, dp0,dp1,dp2,dp3,
> am0,bm0,am1,bm1,am2,bm2,am3,bm3, ap0,bp0,ap1,bp1,ap2,bp2,ap3,bp3,
> ds0,ds1,ds2,ds3, c0,c1, m];
```

5.6.1. *The case $m = 0$.* The following list of abstract energy values is used to internally generate the energy induced order for $m = 0$:

```
> vm_0:=[0,0,0,0, 0.5,0.5,0.5,0.5, 0.5,0.5,0.5,0.5,
> 1,1,1,1,1,1,1,1, 1,1,1,1,1,1,1,1,
> 2,2,2,2, 2.5,2.5, 3];
```

The energy induced order for $m < 0$ (for $m < 0$ the $d_i^{m,-}$'s have lower energy than the $d_i^{m,+}$'s, and similarly $a_i^{m,-}, b_i^{m,-}$ lie below $a_i^{m,+}, b_i^{m,+}$):

```
> vm_neg:=[0,0,0,0, 0.5,0.5,0.5,0.5, 0.7,0.7,0.7,0.7,
> 1,1,1,1,1,1,1,1, 1.5,1.5,1.5,1.5,1.5,1.5,1.5,1.5,
> 2,2,2,2, 2.5,2.5, 3];
```

Again, the energy induced order for $m < 0$ is admissible for $m = 0$, too.

The direct factor $C_2$ acts by $-1$ on the set of functions. It only exists in the $m = 0$ case:

```
> D8C2:=
> [
> [[x0,x1,x2,x3],[dm0,dm1,dm2,dm3],[dp0,dp1,dp2,dp3],
> [am0,am1,am2,am3],[ap0,ap1,ap2,ap3],[bm0,bm1,bm2,bm3],[bp0,bp1,bp2,bp3],
> [ds0,ds1,ds2,ds3],[c0,c1]],
> [[x1,x3],[dm0,dm3],[dm1,dm2],[dp0,dp3],[dp1,dp2],
> [am0,bm3],[am1,bm2],[am2,bm1],[am3,bm0],[ap0,bp3],[ap1,bp2],[ap2,bp1],[ap3,bp0],
> [ds0,ds3],[ds1,ds2],[c0,c1]],
> [[x0,x2],[x1,x3],[dm0,dp2],[dm1,dp3],[dm2,dp0],[dm3,dp1],
> [am0,ap2],[am1,ap3],[am2,ap0],[am3,ap1],[bm0,bp2],[bm1,bp3],[bm2,bp0],[bm3,bp1],
> [ds0,ds2],[ds1,ds3],[c0,c1]]
> ];
```

In the $m \neq 0$ case the symmetry is broken from $D_8 \times C_2$ down to $D_8$:

```
> D8:=D8C2[1..2];
```

The CONLEY indices of the respective MORSE sets (for all $|m|$ small):

```
> CHp:=[0,0,0,0, 0,0,0,0, 0,0,0,0,
> 1,1,1,1,1,1,1,1, 1,1,1,1,1,1,1,1,
> 2,2,2,2 ,2,2, 3];
```

Again, we only add the condition that we are dealing with a global attractor:

```
> CH:=[op(zip((x,y)->x=y,P,CHp)),P=0];
```

Here we discuss the $m = 0$ case. Here we *only* use the homogeneous conditions of [Maier-Paape et al., 2007, Thm. 6.3], i.e. $a = c = d = t = u = v = w = x = y = z = \beta = \gamma = \delta = 0$. Note that the condition $a = 0$ is related to the assumption of a heteroclinic connection from $b_0^{0,-}$ to $x_0^0$ (i.e. $b = 1$, cf. [Maier-Paape et al., 2007, (53)]):

```
> EXTRA_CONDITIONS0:=
> [
> [am0,x0]=0,
> [am1,x0]=0,[bm1,x0]=0,
> [am1,dm0]=0,[bm1,dm0]=0,[am2,dm0]=0,
> [ap0,dm0]=0,[ap1,dm0]=0,[bp1,dm0]=0,[ap2,dm0]=0,
> [ds1,am0]=0,[ds2,am0]=0,[ds3,am0]=0
> ];
```

Our computations

```
> Con0:=ConnectionMatrices(P,vm_0,CH,var,
> "Extra"=EXTRA_CONDITIONS0, "Symmetry"=D8C2):
```

yields two matrices:

$$\left[\left[\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right],\right.$$

$$\left.\left[\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

These two connection matrices are possible connection matrices for $m = 0$ in the range $\lambda^*_{3,0} < \lambda < \lambda^*_{4,0}$. The result agrees with [Maier-Paape et al., 2007, Thm. 6.3]. Note that our computation shows that one doesn't need to impose any of the affine conditions stated in the Theorem!

Now we compute the transition matrices for $m = 0$ and variable $\lambda$ with $\lambda^*_{3,0} < \lambda < \lambda^*_{4,0}$. Using

```
> Ts0:=TransitionMatricesGenerators(P,vm_0,CHp,var,"Symmetry"=D8C2):
```

we obtain three generators, but according to [Maier-Paape et al., 2007, Proof of Thm. 6.3] the first two transition matrices can be ruled out by an easy attractor argument. The remaining transition matrix

$$\left[\mathrm{Id}_{12\times 12}, \mathrm{Id}_{16\times 16}, \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}, \mathrm{Id}_{1\times 1}\right]$$

permutes the above two connection matrices in `Con0`.

5.6.2. *Preparing the transition from $m = 0$ to $m < 0$.* Here we intend to use the algebraic argument introduced in Subsection 5.4.4 to filter out the relevant canonical generators of the transition matrix group, without further use of dynamical arguments. In order to do so we give several bifurcation intervals:

The following set $J_1$ containing index 1 and index 2 equilibria is an energy induced bifurcation interval (i.e. not an interval in the energy induced order for $m < 0$, but one for $m = 0$). Hence in the flow induced order it remains an interval for $m < 0$ small:

```
> J1:=[am0,bm0,am1,bm1,am2,bm2,am3,bm3, ds0,ds1,ds2,ds3];
```

The groups $H_*\Delta(J_1)$ coincide for both possible connection matrices $\Delta$ in `Con0` and are equal to $(0, (\mathbb{Z}/2\mathbb{Z})^4, 0, 0, \ldots)$. Hence, $CH_*(M(J_1)) = (0, (\mathbb{Z}/2\mathbb{Z})^4, 0, 0, \ldots)$.

The following set $J_2$ containing index 0 and index 1 equilibria is also a bifurcation interval from $m = 0$ to $m < 0$, which again remains an interval in the flow induced order for $m < 0$ *small*:

```
> J2:=[dm0,dm1,dm2,dm3, ap0,bp0,ap1,bp1,ap2,bp2,ap3,bp3];
```

Again, the groups $H_*\Delta(J_2)$ coincide for both possible connection matrices in `Con0` and are equal to $((\mathbb{Z}/2\mathbb{Z})^4, (\mathbb{Z}/2\mathbb{Z})^8, 0, 0, \ldots)$. Hence, $CH_*(M(J_2)) = ((\mathbb{Z}/2\mathbb{Z})^4, (\mathbb{Z}/2\mathbb{Z})^8, 0, 0, \ldots)$.

The same is true for the following interval $J_3$ and the CONLEY index of the isolated invariant set $M(J_3)$ is $(0, (\mathbb{Z}/2\mathbb{Z})^4, 0, 0, \ldots)$:

```
> J3:=[dm0,dm1,dm2,dm3, am0,bm0,am1,bm1,am2,bm2,am3,bm3];
```

Finally, the same applies to the following set $J_4$ and the CONLEY index of $M(J_4)$ is $(0, (\mathbb{Z}/2\mathbb{Z})^4, 0, 0, \ldots)$:

```
> J4:=[dp0,dp1,dp2,dp3, ap0,bp0,ap1,bp1,ap2,bp2,ap3,bp3];
```

Our goal is to find connection matrices for $m < 0$ small satisfying the extra index conditions imposed by the above four intervals:

```
> CH_J := [ op(CH), J1=[ [1], [[0, 0, 0, 0]] ],
> J2=[ [[0, 0, 0, 0]], [[0, 0, 0, 0, 0, 0, 0, 0]] ],
> J3=[ [1], [[0, 0, 0, 0]] ], J4 = [ [1], [[0, 0, 0, 0]] ] ];
```

The syntax in `J1=[ [1], [[0, 0, 0, 0]] ]` is derived from

$$CH_*(M(J_1)) = (0 = (\mathbb{Z}/2\mathbb{Z})/1(\mathbb{Z}/2\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})^4 = \bigoplus_{i=1,\ldots,4} (\mathbb{Z}/2\mathbb{Z})/0(\mathbb{Z}/2\mathbb{Z}), 0, 0, \ldots).$$

5.6.3. *The case $m < 0$.* In the following we compute the transition matrices for the passage from $m = 0$ with symmetry group $D_8 \times C_2$ to $m < 0$ small with symmetry group $D_8$, excluding those canonical generators, that take connection matrices in `Con0` at $m = 0$ to those violating the Conley index conditions $CH_*(M(J))$ of the Morse sets $M(J)$ for $J \in J_1, J_2, J_3, J_4$ (note that the extra conditions are only used to calculate `Con0`):

```
> obj:=P,vm_0,vm_neg,CHp,CH,CH_J,var,
> "Symmetry"=D8C2,"Symmetry"=D8,"Extra"=EXTRA_CONDITIONS0:
> G:=TransitionMatricesOfBranching(obj);
```

We obtain 6 canonical generators for the transition matrix group. Originally 17 canonical generators were computed and 11 have been ruled out, by the additional interval conditions stated above. Internally the procedure `TransitionMatricesOfBranching` computed the connection matrices for $m = 0$ with the large symmetry group $D_8 \times C_2$. After that it computed the canonical generators of the transition matrix group for the passage from $m = 0$ to $m < 0$ small (the symmetry is broken to $D_8$). Then it applied each such generator to the $m = 0$ connection matrices and checked, whether the complete orbit satisfies the additional index conditions for the intervals $J_1, \ldots, J_4$. Additionally, it can be shown by leaving out the conditions for the intervals $J_3$ and $J_4$ that $J_1$ and $J_2$ suffice to obtain the six matrices.

The transition matrix group is elementary abelian of order 64. Applying the transition matrix group $G$ to the two connection matrices at $m = 0$ we get a transitive set containing 64 connection matrices, which constitute possible connection matrices for $m < 0$ small:

```
> Orbits(G,Con0,var,"conjugation"):
```

The above two steps can finally be summed up in the procedure `ConnectionMatricesOf-Branching` with the same input as `TransitionMatricesOfBranching`:

```
> ConB:=ConnectionMatricesOfBranching(obj);
```

We suggest further numerical studies to verify specific heteroclinic connections, in order to eventually decide which of the 64 so far obtained possible connection matrices is the *one*.

## References

[Barakat et al., 2008] Barakat, M., Maier-Paape, S., and Robertz, D. (2006-2008). `conley` *project*. (http://wwwb.math.rwth-aachen.de/conley). 3, 12, 17, 20, 23, 24

[Barakat and Robertz, 2009] Barakat, M. and Robertz, D. (2009). `conley` – computing connection matrices in Maple. *J. Symb. Comp.*, 44:540–557. (arXiv:math.DS/0701173). 2, 8, 9, 10, 22, 23

[Conley, 1978] Conley, C. (1978). *Isolated invariant sets and the Morse index*, volume 38 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I. 6

[Franzosa, 1986] Franzosa, R. D. (1986). Index filtrations and the homology index braid for partially ordered Morse decompositions. *Trans. Amer. Math. Soc.*, 298(1):193–213. 10

[Franzosa, 1988] Franzosa, R. D. (1988). The continuation theory for Morse decompositions and connection matrices. *Trans. Amer. Math. Soc.*, 310(2):781–803. 8, 10

[Franzosa, 1989] Franzosa, R. D. (1989). The connection matrix theory for Morse decompositions. *Trans. Amer. Math. Soc.*, 311(2):561–592. 2, 3, 5, 8, 9, 10, 11, 14, 15, 16, 18, 19, 30

[Franzosa and Mischaikow, 1988] Franzosa, R. D. and Mischaikow, K. (1988). The connection matrix theory for semiflows on (not necessarily locally compact) metric spaces. *J. Differential Equations*, 71(2):270–287. 10

[Franzosa and Mischaikow, 1998] Franzosa, R. D. and Mischaikow, K. (1998). Algebraic transition matrices in the Conley index theory. *Trans. Amer. Math. Soc.*, 350(3):889–912. 27

[Maier-Paape et al., 2008] Maier-Paape, S., Miller, U., and Wanner, T. (2008). Secondary bifurcation for the Cahn-Hilliard equation on a square. in preparation. 20, 31

[Maier-Paape et al., 2007] Maier-Paape, S., Mischaikow, K., and Wanner, T. (2007). Structure of the attractor of the Cahn-Hilliard equation on a square. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 17(4):1221–1263. 1, 2, 11, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33

[McCord, 1988] McCord, C. (1988). The connection map for attractor-repeller pairs. *Trans. Amer. Math. Soc.*, 307(1):195–203. 27

[McCord and Mischaikow, 1996] McCord, C. and Mischaikow, K. (1996). On the global dynamics of attractors for scalar delay equations. *J. Amer. Math. Soc.*, 9(4):1095–1133. 6

[Mischaikow, 1995] Mischaikow, K. (1995). Conley index theory. In *Dynamical systems (Montecatini Terme, 1994)*, volume 1609 of *Lecture Notes in Math.*, pages 119–207. Springer, Berlin. 10

[Mischaikow and Mrozek, 2002] Mischaikow, K. and Mrozek, M. (2002). Conley index. In *Handbook of dynamical systems, Vol. 2*, pages 393–460. North-Holland, Amsterdam. 2, 6, 8, 9, 14, 27, 29

[Salamon, 1985] Salamon, D. (1985). Connected simple systems and the Conley index of isolated invariant sets. *Trans. Amer. Math. Soc.*, 291(1):1–41. 10

Lehrstuhl B für Mathematik, RWTH-Aachen University, 52062 Germany
*E-mail address*: mohamed.barakat@rwth-aachen.de

Institut für Mathematik, RWTH-Aachen University, 52062 Germany
*E-mail address*: maier@instmath.rwth-aachen.de