Simplicial Cohomology of Smooth Orbifolds in GAP

Mohamed Barakat[1] and Simon Görtzen[2]

[1] Department of mathematics, University of Kaiserslautern, 67653 Kaiserslautern, Germany, barakat@mathematik.uni-kl.de

[2] UMIC Research Centre, RWTH Aachen, Mies-van-der-Rohe-Str. 15, 52074 Aachen, Germany, simon.goertzen@rwth-aachen.de

Abstract. This short research announcement briefly describes the simplicial method underlying the GAP package SCO for computing the so-called orbifold cohomology of topological resp. smooth orbifolds. SCO can be used to compute the lower dimensional group cohomology of some infinite groups.

Instead of giving a complete formal definition of an orbifold, we start with a simple construction, general enough to give rise to any orbifold $\mathcal{M}$. Let X be a (smooth) manifold and G a Lie group acting (smoothly and) properly on X, i.e., the action graph $\alpha : G \times X \to X \times X : \ (g, x) \mapsto (x, gx)$ is a proper map. In particular, G acts with *compact* stabilizers $G_x = \alpha^{-1}(\{(x, x)\})$. Further assume that G is either

- discrete (acting discontinuously), or
- compact acting almost freely (i.e. with discrete stabilizers) on X.

In both cases G acts with *finite* stabilizers. "Enriching" $M := X/G$ with these finite isotropy groups leads to the so-called **fine** orbit space $\mathcal{M} := [X/G]$, also called the **global quotient orbifold**. We call the orbifold **reduced** if the action is faithful. The topological space M is then called the **coarse space** underlying the orbifold $\mathcal{M}$. Roughly speaking, a reduced **orbifold** $\mathcal{M}$ locally looks like the orbit space $\mathbb{R}^n/V$, where V is a *finite* subgroup of $\mathrm{GL}_n(\mathbb{R})$. It is easy to show that every effective orbifold $\mathcal{M}$ arises as a global quotient orbifold by a compact Lie group[3] G.

By considering (countable) discrete groups G acting properly and discontinuously on X we still obtain a subclass of orbifolds which includes many interesting moduli spaces. The most prominent one is the (non-compactified) moduli space $\mathcal{M}_{g,n}$ of curves of genus g with n marked points and $2g + n \geq 3$. It is the global quotient $[T_{g,n}/\Gamma_{g,n}]$ of the *contractible* Teichmüller space $T_{g,n} \approx \mathbb{C}^{3g-3+n}$ by the proper discontinuous action of the **mapping class group** $\Gamma_{g,n}$ [7].

[3] Define X as the bundle of orthonormal frames on $\mathcal{M}$. It is a manifold on which the orthogonal group $G := \mathrm{O}_n(\mathbb{R})$ acts almost freely with faithful slice representations [8, Thm. 4.1].

A convenient way to define a cohomology theory on $\mathcal{M}$ is to consider the category $\mathrm{Ab}(\mathcal{M})$ of ABELian sheaves on $\mathcal{M}$, together with the global section functor $\Gamma_{\mathcal{M}} : \mathrm{Ab}(\mathcal{M}) \to \mathrm{Ab}$ to the category of ABELian groups. The ABELian category $\mathrm{Ab}(\mathcal{M})$ has enough injectives and the orbifold cohomology of $\mathcal{M}$ with values in a sheaf[4] $\mathcal{A}$ is then simply defined as the derived functor cohomology

$$H^n(\mathcal{M}, \mathcal{A}) := \mathrm{R}^n \Gamma_{\mathcal{M}}(\mathcal{A}).$$

This conceptually simple definition is of course highly nonconstructive. To boil it down to a constructive one we proceed in several steps following [9]:

1. If the orbifold $\mathcal{M}$ is given as a global quotient $[X/G]$, then $\mathrm{Ab}(\mathcal{M})$ is equivalent to the category $\mathrm{Ab}(X)^G$ of G-equivariant sheaves on X. The global section functor $\Gamma_{\mathcal{M}}$ then corresponds to the functor Γ_X^G of G-equivariant global sections. The space X and the action graph $\alpha : G \times X \to X \times X$ are encoded by the so-called **action groupoid**[5] $\mathcal{G} = G \ltimes X := G \times X \rightrightarrows X$ with source-target map $(s,t) = \alpha$, composition $(g,y)(h,x) = (ghx, x)$ for $y = hx$, and inversion $(g,x)^{-1} = (g^{-1}, gx)$.
   For a general groupoid $\mathcal{G}$ with base X denote by $\mathrm{Ab}(\mathcal{G})$ the ABELian category of $\mathcal{G}$-sheaves, i.e., the sheaves on X compatible with the action of the groupoid $\mathcal{G}$. Hence, if $\mathcal{G}$ is the action groupoid $G \ltimes X$, then the $\mathcal{G}$-sheaves are nothing but G-equivariant sheaves on X and $\mathrm{Ab}(\mathcal{G}) = \mathrm{Ab}(X)^G$.
2. Another way to represent an orbifold $\mathcal{M}$ was suggested by HAEFLIGER [6]. Denoting by $\mathcal{U}$ an orbifold atlas of $\mathcal{M}$, he constructed an étale proper groupoid $H := H_1 \rightrightarrows H_0$ with base space $H_0 := \coprod_{\tilde{U} \in \mathcal{U}} \tilde{U}$ and orbit space[6] H_0/H_1 homeomorphic[7] to the coarse space M underlying $\mathcal{M}$. Again it follows that $\mathrm{Ab}(H) \simeq \mathrm{Ab}(\mathcal{M})$.
3. Choose a triangulation $\mathcal{T}$ of M **adapted** to the orbifold atlas $\mathcal{U}$ of $\mathcal{M}$. Such a triangulation always exists [9, Prop. 1.2.1]. Replacing H_0 by the subspace $R_0(\mathcal{T}) := \coprod_{\sigma \in \mathcal{T}_n} \widetilde{\sigma} \subset H_0$, where $\mathcal{T}_n$ is the set of maximal simplices in $\mathcal{T}$ (of dimension $n = \dim \mathcal{M}$), and pulling back H_1 over the embedding $R_0(\mathcal{T}) \times R_0(\mathcal{T}) \hookrightarrow H_0 \times H_0$ yields the set of arrows $R_1(\mathcal{T})$ of the so-called **reduced groupoid** $R(\mathcal{T}) := R_1(\mathcal{T}) \rightrightarrows R_0(\mathcal{T})$. $R(\mathcal{T})$ is a full subgroupoid of H and, again, $\mathrm{Ab}(R(\mathcal{T})) \simeq \mathrm{Ab}(H)$. In particular $H^n(R(\mathcal{T}), A_{|R(\mathcal{T})}) \cong H^n(H, A)$ for any ABELian sheaf A on H.
4. One can now show that the connected components of $R(\mathcal{T})_\bullet$ are contractible, where $R(\mathcal{T})_\bullet$ denotes the nerve of the reduced groupoid $R(\mathcal{T})$. The set

   $$S_\bullet := \pi_0(R(\mathcal{T})_\bullet)$$

   of connected components of the nerve of the reduced groupoid is a simplicial set, which is, unlike $R(\mathcal{T})_\bullet$, *not* a nerve of some category, in general.

[4] One often considers the constant sheaf $\mathcal{A} = \mathbb{Z}$.

[5] The group G cannot be recovered from the action groupoid, in general.

[6] This is the space of equivalence classes of the equivalence relation given by the image of the source-target map $(s,t) : H_1 \to H_0 \times H_0$.

[7] This generalizes a similar construction for an étale proper groupoid representing a manifold, cf. [2, Chap. 2, 2.α]

5. A local system $\mathcal{A}$ on $\mathcal{M}$, i.e., a locally constant sheaf, induces a locally constant sheaf on H and by restriction one on $R(\mathcal{T})$. The induced locally constant sheaf $A^{(\bullet)}$ on the nerve[8] $R(\mathcal{T})_\bullet$ is constant on the contractible connected components and factors to a local system of coefficients (cf. [4, I.7]) on the simplicial set $S_\bullet$, which we again denote by A.
6. A spectral sequence argument finally provides the isomorphism of cohomology:
$$H^n(\mathcal{M}, \mathcal{A}) \cong H^n(S_\bullet, A).$$

The right hand side of this isomorphism is indeed constructive and was implemented in the GAP package SCO [5]. The package includes examples computing the low dimensional cohomology groups of the 2-dimensional crystallographic space groups:
In case $\mathcal{M}$ is a global quotient $[X/G]$ of a *contractible* space X by a proper action of a group G, then $H^n(\mathcal{M}, \mathbb{Z}) \cong H^n(G, \mathbb{Z})$, and we recover the ordinary group cohomology of G.

In other words: The cohomology functor cannot detect the passage from an abstract group G (regarded as a groupoid over one point) to the action groupoid $G \ltimes X$ on a contractible space X (as a contractible space does not introduce any detectable cohomological data). In favorable situations this can be exploited to replace an infinite (discrete) group by an orbifold, an object which is still group- but now also *space*-like, and where the occurring groups are all *finite*. So, roughly speaking, the cohomology of an infinite[9] (discrete) group can be computed as the cohomology of a space "intertwined" with the cohomology of some finite groups.

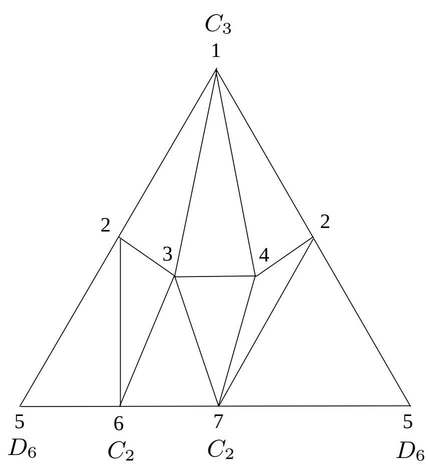

Fig. 1. Adapted triangulation of the fundamental domain of `p31m`.

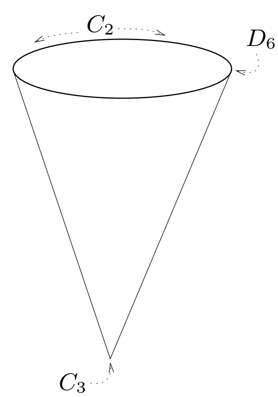

Fig. 2. The coarse space of the orbifold `p31m` enriched with the isotropy groups.

[8] In the sense of [3].
[9] Starting with a finite group only produces computational overhead.

Starting from a fundamental domain of a proper discontinuous cocompact action of a discrete group one can easily obtain an adapted triangulation of the global quotient. This was the point of departure for computing the lower dimensional cohomology of the 2-dimensional space groups, which can easily be extended to higher dimensional space groups. The SCO package can be used to quickly produce the simplicial set $S_\bullet$ associated to an adapted triangulation, but the efficiency of computing the cohomology will depend on the growth of the cardinalities $|S_n|$, where S_n is n-th set of $S_\bullet$. It is thus desirable to construct "minimal" triangulations.

The orbifold corresponding to the 2-dimensional space group p31m is a contractible cone with a singular C_3-point at the bottom, as well as a C_2-edge with one D_6-isotropy (see Figures 1 and 2). Just as with D_6 itself, this leads to a 4-periodic cohomology:

$$H^i(\mathtt{p31m}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & i \geq 2,\ i \equiv_4 2 \\ \mathbb{Z}/2\mathbb{Z} & i \geq 3,\ i \equiv_4 1, 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2 & i \geq 4,\ i \equiv_4 0 \end{cases}$$

More details are provided in the forthcoming work [1].

References

1. Barakat, M., Görtzen, S.: Simplicial cohomology of orbifolds revisited. In preparation 4
2. Connes, A.: Noncommutative geometry. Academic Press Inc., San Diego, CA (1994), (http://www.alainconnes.org/downloads.html) 2
3. Deligne, P.: Théorie de Hodge. III. Inst. Hautes Études Sci. Publ. Math. (44), 5–77 (1974) 3
4. Gelfand, S.I., Manin, Y.I.: Methods of homological algebra. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2. edn. (2003) 3
5. Görtzen, S.: GAP Package SCO (2007-2008), http://wwwb.math.rwth-aachen.de/goertzen/SCO 3
6. Haefliger, A.: Groupoïdes d'holonomie et classifiants. Astérisque (116), 70–97 (1984), transversal structure of foliations (Toulouse, 1982) 2
7. Korkmaz, M.: Low-dimensional homology groups of mapping class groups: a survey. Turkish J. Math. 26(1), 101–114 (2002), (arXiv:math.GT/0307111) 1
8. Moerdijk, I., Pronk, D.A.: Orbifolds, sheaves and groupoids. K-Theory 12(1), 3–21 (1997) 1
9. Moerdijk, I., Pronk, D.A.: Simplicial cohomology of orbifolds. Indag. Math. (N.S.) 10(2), 269–293 (1999), (arXiv:q-alg/9708021) 2