Jets. A Maple-Package for Formal Differential Geometry

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Abstract. The Maple-package `jets` was first designed to be an extension of the package `desolv`. In the current stage it became an independent package going beyond symmetries to handle different aspects of formal differential geometry, including some important parts of the variational bicomplex. We demonstrate this by computing the set of all Hamiltonian structures of a order at most 3, which are compatible with D_x. This set includes among others the famous KdV-operator $D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$.

1 Introduction

The Maple-package `jets`, originally an extension of the package `desolv`[1] adding to it the facility of computing generalized symmetries of differential equations, is at the current stage an independent package going beyond symmetries to handle different aspects of what I. M. Gel'fand, in his 1970 address to the International Congress in Nice, called "formal differential geometry". Important parts of the variational bicomplex, as playing a crucial role in the formal theory, are implemented in `jets`. Most of the implementation of the variational aspects in `jets`, such as variational symmetries, higher Euler operators, homotopy operators and conservation laws, was done by Gehrt Hartjen as part of his diploma thesis [Har]. As dual to functional forms and the vertical derivative also functional multi-vectors and the Nijenhuis-Schouten bracket are also implemented in `jets`, enabling one to handle Hamiltonian systems of evolution equations and nonlinear integrable systems. The package adds to Maple the important feature of dealing with jet calculus, a thing which is still missing in modern computer algebra systems. Almost every formula appearing in [Olv] can now be computed using `jets`.

2 Hamiltonian Structures and the Nijenhuis-Schouten Bracket

As mentioned in the abstract, the aim of this paper is to demonstrate a non-trivial application of the package `jets` by computing the set of all Hamiltonian

[1] `desolv` was written by Khai Vu and Colin McIntosh. `jets` still uses `desolv` to solve linear PDE systems.

structures of a order at most 3, which are compatible with D_x. This is done in section 3. To this end we define the notion of functional multi-vectors, Hamiltonian structures and the Nijenhuis-Schouten bracket. The notions used in sequel are standard and can be found in [Olv]. Further details are found in [Bar].

Let $E \longrightarrow M$ be a fibred manifold in p independent variables $(x^i) = (x^1, \ldots, x^p)$ and q dependent variables $(u^\alpha) = (u^1, \ldots, u^q)$. By $J_\infty(E) \rightarrow M$ we denote the infinite jet bundle having the jet variables (x^i, u^α_J) as coordinates, where J is an arbitrary multi-index. By $\mathcal{A}$ we denote the space of differential expressions over E, i.e. smooth real-valued functions of finitely many arbitrary jet variables. By $\mathcal{V}^1$ we denote the space of evolutionary vector fields, or equivalently the space of characteristics over a jet bundle. This space can be identified with the Cartesian power $\mathcal{A}^q$. Further we define locally $\mathcal{F}^0 := \mathcal{A}/\mathsf{Div}(\mathcal{A}^p)$ and call it the space of functionals[2]. By $\mathcal{F}^1$ we denote the $\mathcal{F}^0$-dual space of $\mathcal{V}^1$. We can also identify it with $\mathcal{A}^q$. Further let $\mathcal{F}^n$ (resp. $\mathcal{V}^n$) denote the space of functional n-forms (resp. n-vectors).

We first note the following two basic formulas. The first one relates the prolongation of an evolutionary vector field and the Fréchet derivative

$$\mathsf{pr}\,\mathbf{v}_Q(L) = \mathsf{D}_L Q, \tag{1}$$

where $Q = (Q^1, \ldots, Q^q)^{\mathrm{tr}}$ is a characteristic, $\mathbf{v}_Q = Q^\alpha \frac{\partial}{\partial u^\alpha}$ and evolutionary vector field, $\mathsf{pr}\,\mathbf{v}_Q = D_J Q^\alpha \frac{\partial}{\partial u^\alpha_J}$ (prolongation formula) and $\mathsf{D}_L = (\frac{\partial L}{\partial u^1_J} D_J, \ldots, \frac{\partial L}{\partial u^q_J} D_J)$ (Fréchet derivative). The proof follows immediately from the prolongation formula and the definition of the Fréchet derivative. The second formula is the standard Leibniz rule

$$\mathsf{pr}\,\mathbf{v}(L \cdot P) = \mathsf{pr}\,\mathbf{v}L \cdot P + L \cdot \mathsf{pr}\,\mathbf{v}P \tag{2}$$

where $\mathbf{v}$ is a generalized vector field and L, P are arbitrary differential expression.

We still need the following lemma.

Lemma 1. *For a differential operator $\mathcal{D} = P^J D_J$ ($P^J \in \mathcal{A}$) and differential function $T \in \mathcal{A}$, we have the following Leibniz rule:*

$$\mathsf{pr}\,\mathbf{v}_Q(\mathcal{D}T) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{pr}\,\mathbf{v}_Q(T), \tag{3}$$

or equivalently by (1)

$$\mathsf{D}_{\mathcal{D}T}(Q) = \mathsf{pr}\,\mathbf{v}_Q(\mathcal{D})T + \mathcal{D}\mathsf{D}_T Q. \tag{4}$$

Proof. [Olv], Formula (5.38). □

Definition 1 (Adjoint operator). *The formal adjoint operator of a matrix differential operator $\mathcal{D} = (P^J_{\alpha\beta} D_J)$ is defined by*

$$\mathcal{D}^* = ((-1)^{|J|} D_J P^J_{\beta\alpha}).$$

[2] $\mathsf{Div}P = D_i P^i$, where $P = (P^1, \ldots, P^p)$ and $D_i = D_{x^i}$

Definition 2 (Euler operator). *For $L \in \mathcal{A}$ the operator*

$$\mathsf{E}(L) := \mathsf{D}_L^*(1) \tag{5}$$

is called the Euler *operator.*

Lemma 2 ([Olv], Formula (4.15)). *A Lagrangian $L \in \mathcal{A}$ transforms infinitesimally according to the rule*

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\,\mathbf{v} L + L \mathsf{Div}(\xi), \tag{6}$$

where $\mathbf{v} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ is a generalized vector field[3].

Proof. [Olv], Theorem 4.12. □

Corollary 1 (Lie derivative of functionals). *For a Lagrangian L viewed as an element of $\mathcal{F}^0$, i.e. as a functional 0-form, the Lie derivative $\mathcal{L}_{\mathbf{v}}$ satisfies*

$$\mathcal{L}_{\mathbf{v}} L = \mathsf{pr}\,\mathbf{v}_Q L = \mathsf{E}(L) \cdot Q. \tag{7}$$

Proof. The following are identities between functionals. For a generalized vector field $\mathbf{v}$ with characteristic Q

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}} L &\overset{(6)}{=} \mathsf{pr}\,\mathbf{v} L + L \mathsf{Div}(\xi) \\
&= \mathsf{pr}\,\mathbf{v}_Q L + \xi^i D_i L + L D_i \xi^i \\
&= \mathsf{pr}\,\mathbf{v}_Q L + \mathsf{Div}(L\xi) \\
&= \mathsf{pr}\,\mathbf{v}_Q L \\
&\overset{(1)}{=} 1 \cdot \mathsf{D}_L(Q) \\
&= \mathsf{D}_L^*(1) \cdot Q \\
&\overset{(5)}{=} \mathsf{E}(L) \cdot Q
\end{aligned}$$

□

Definition 3 (Lie derivative of vector fields). *Let $\mathbf{v}$ be a generalized vector field and R a characteristic, i.e. $R \in \mathcal{V}^1$. Define the Lie derivative of R with respect to $\mathbf{v}$ by*

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}(R) &= \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{pr}\,\mathbf{v}_R Q \\
&\overset{(1)}{=} \mathsf{pr}\,\mathbf{v}_Q R - \mathsf{D}_Q R,
\end{aligned} \tag{8}$$

where Q is the characteristic of $\mathbf{v}$.

[3] [Olv] proves this for point vector fields only. The above Lie derivative coincides with the notion of projected Lie derivative $\mathcal{L}_{\mathbf{v}}^\sharp$ introduced in [And], Chapter 3.

Proposition 1 (Lie derivative of functional 1-forms). *For the Lie derivative of a source form $\Delta \in \mathcal{F}^1$ the following two statements are equivalent:*

(i) *Δ transforms infinitesimally according to*

$$\mathcal{L}_{\mathbf{v}_Q} \Delta = \mathsf{pr}\, \mathbf{v}_Q \Delta + \mathsf{D}_Q^* \Delta. \tag{9}$$

(ii) *$\mathcal{L}_{\mathbf{v}_Q}$ satisfies the following* Leibniz rule *for an arbitrary characteristic R*

$$\mathcal{L}_{\mathbf{v}_Q}(\Delta \cdot R) = \mathcal{L}_{\mathbf{v}_Q} \Delta \cdot R + \Delta \cdot \mathcal{L}_{\mathbf{v}_Q} R. \tag{10}$$

This is an identity of functionals, i.e. the left and right hand sides are equal up to local divergence.

Proof. Both directions follow from the following equalities:

$$\begin{aligned}
&\mathsf{E}(\mathcal{L}_{\mathbf{v}_Q}(\Delta \cdot R)) - \mathsf{E}(\Delta \cdot \mathcal{L}_{\mathbf{v}_Q} R) \\
&\overset{(7)}{=} \mathsf{E}(\mathsf{pr}\, \mathbf{v}_Q(\Delta \cdot R)) - \mathsf{E}(\Delta \cdot \mathcal{L}_{\mathbf{v}_Q} R) \\
&\overset{(2),(8)}{=} \mathsf{E}(\mathsf{pr}\, \mathbf{v}_Q \Delta \cdot R + \Delta \cdot \mathsf{pr}\, \mathbf{v}_Q R) - \mathsf{E}(\Delta \cdot (\mathsf{pr}\, \mathbf{v}_Q R - \mathsf{D}_Q R)) \\
&= \mathsf{E}(\mathsf{pr}\, \mathbf{v}_Q \Delta \cdot R + \Delta \cdot \mathsf{D}_Q R) \\
&= \mathsf{E}((\mathsf{pr}\, \mathbf{v}_Q \Delta + \mathsf{D}_Q^* \Delta) \cdot R).
\end{aligned}$$

□

Remark 1. The identity of functionals

$$\mathcal{L}_{\mathbf{v}_Q} \Delta \cdot R = \mathsf{pr}\, \mathbf{v}_Q \Delta \cdot R + \Delta \cdot \mathsf{pr}\, \mathbf{v}_R Q, \tag{11}$$

which is part of the proof, appears as formula (4.2) in [GDo2].

Lemma 3. *The following identity holds for a general $\mathcal{K} : \mathcal{F}^1 \to \mathcal{V}^1$*

$$(\mathsf{pr}\, \mathbf{v}_{.}(\mathcal{K})\Delta)^* \Sigma = (\mathsf{pr}\, \mathbf{v}_{.}(\mathcal{K}^*)\Sigma)^* \Delta \tag{12}$$

Proof. For an arbitrary characteristic S

$$\begin{aligned}
&\mathsf{E}(S \cdot ((\mathsf{pr}\, \mathbf{v}_{.}(\mathcal{K})\Delta)^* \Sigma - (\mathsf{pr}\, \mathbf{v}_{.}(\mathcal{K}^*)\Sigma)^* \Delta)) \\
&= \mathsf{E}(\mathsf{pr}\, \mathbf{v}_S(\mathcal{K})\Delta \cdot \Sigma - \mathsf{pr}\, \mathbf{v}_S(\mathcal{K}^*)\Sigma \cdot \Delta) \\
&= \mathsf{E}(\mathsf{pr}\, \mathbf{v}_S(\mathcal{K})\Delta \cdot \Sigma - \Sigma \cdot \mathsf{pr}\, \mathbf{v}_S(\mathcal{K})\Delta) \\
&= 0.
\end{aligned}$$

□

Definition 4 (Nijenhuis-Schouten bracket). *For $\mathcal{D},\mathcal{E}\in\mathcal{V}^2$ the Nijenhuis-Schouten bracket $[\mathcal{D},\mathcal{E}]:\mathcal{F}^1\times\mathcal{F}^1\times\mathcal{F}^1\to\mathcal{F}^0$ is defined as follows:*

$$[\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3):=\mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3+\mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3+(\text{cycle}),\tag{13}$$

where the word (cycle) means summation over all cyclic permutations of the indices $1,2,3$. $\mathcal{D}$ and $\mathcal{E}$ are viewed as differential operators from $\mathcal{F}^1$ into $\mathcal{V}^1$.

This definition is a generalisation of the classical Nijenhuis-Schouten bracket from differential geometry, which is one of its advantages. It appears in [GDo2], Formula (3.3). Nevertheless there are two major *drawbacks* of this definition. The first one is that the right hand side is a functional, so it has no normal form. This means that checking the vanishing of the bracket or extracting conditions for its vanishing is not a direct procedure. The second one is that one needs more than total differentials of the Δ_i's, meaning that we cannot compute with general Δ_i's, complicating the check of vanishing of the bracket. Besides, from this definition we do not see that the bracket of two 2-vectors is a $(3,0)$-tensor, *even* a 3-vector. In the following we want to make use of the freedom of adding divergences to circumvent these drawbacks. The following formula cures both drawbacks.

Proposition 2 (Nijenhuis-Schouten bracket). *For $\mathcal{D},\mathcal{E}\in\mathcal{V}^2$ the following formula is an equivalent definition of the Nijenhuis-Schouten bracket $[\mathcal{D},\mathcal{E}]$*

$$\begin{aligned}[\mathcal{D},\mathcal{E}](\Delta)=\;&\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta}(\mathcal{E})-\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta+(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta)^*+\\&\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta}(\mathcal{D})-\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta+(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta)^*.\end{aligned}\tag{14}$$

Proof.

$$\begin{aligned}
&\mathsf{E}([\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3))\\
&=\mathsf{E}(\mathcal{L}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\mathcal{L}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+(\text{cycle})\\
&\overset{(11)}{=}\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D}\Delta_1))+\\
&\quad\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E}\Delta_1))+\\
&\quad(\text{cycle})\\
&\overset{(3)}{=}\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)+\mathsf{E}(\Delta_2\cdot\mathcal{D}\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1)+\\
&\quad\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)+\mathsf{E}(\Delta_2\cdot\mathcal{E}\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1)+\\
&\quad(\text{cycle})\\
&=\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)-\mathsf{E}(\mathcal{D}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1)+\\
&\quad\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)-\mathsf{E}(\mathcal{E}\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1)+\\
&\quad(\text{cycle})\\
&=\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}\Delta_2\cdot\mathcal{E}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)-\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}\Delta_1\cdot\mathcal{D}\Delta_2)+\\
&\quad\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}\Delta_2\cdot\mathcal{D}\Delta_3)+\mathsf{E}(\Delta_2\cdot\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1)-\mathsf{E}(\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}\Delta_1\cdot\mathcal{E}\Delta_2)+
\end{aligned}$$

$$
\begin{aligned}
&\text{(cycle)}\\
\overset{\text{(cycle)}}{=}\;& \mathsf{E}(\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1\\
& +\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1)\\
=\;& \mathsf{E}(\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 - \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_1 \cdot \Delta_3 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^*\Delta_2 \cdot \Delta_3\\
& +\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 - \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_1 \cdot \Delta_3 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*\Delta_2 \cdot \Delta_3)\\
=\;& \mathsf{E}(\Delta_3 \cdot (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta_1)^*\\
& \qquad +\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D}) - \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1 + (\mathsf{pr}\,\mathbf{v}_{\mathcal{E}\cdot}(\mathcal{D})\Delta_1)^*)\Delta_2).
\end{aligned}
$$

□

Remark 2. The right hand side of the formula

$$
\begin{aligned}
&[\mathcal{D},\mathcal{E}](\Delta_1,\Delta_2,\Delta_3)\\
&= \Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_1}(\mathcal{E})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_2}(\mathcal{E})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{D}\Delta_3}(\mathcal{E})\Delta_1\\
&\quad +\Delta_3 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_1}(\mathcal{D})\Delta_2 + \Delta_1 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_2}(\mathcal{D})\Delta_3 + \Delta_2 \cdot \mathsf{pr}\,\mathbf{v}_{\mathcal{E}\Delta_3}(\mathcal{D})\Delta_1,
\end{aligned}
\tag{15}
$$

which is part of the proof, appears as formula (7.30) in [Olv]. This formula is an identity of functionals. This definition still has the first drawback, that trivial functionals do not in general vanish identically, but only up to local divergence. The second drawback is eliminated and one can see the $(3,0)$-tensoriality of the expression. But due to the first drawback it still *not* completely easy to see that this expression is in fact a 3-vector. If we instead use Proposition 2 to define the bracket, these properties follow immediately:

Lemma 4. *The Nijenhuis-Schouten bracket satisfies the following properties:*

(i) $[\mathcal{D},\mathcal{E}]$ *is a* 3*-vector, i.e. is totally skew-adjoint:*
   (a) $[\mathcal{D},\mathcal{E}](\Delta)$ *is a total differential operator in the source form* Δ*.*
   (b) $[\mathcal{D},\mathcal{E}](\Delta)$ *is skew-adjoint.*
   (c) $[\mathcal{D},\mathcal{E}](\Delta)\Sigma = -[\mathcal{D},\mathcal{E}](\Sigma)\Delta$*.*
(ii) $[\mathcal{D},\mathcal{E}] = [\mathcal{E},\mathcal{D}]$*.*

Proof. (i.b) follows immediately from the skew-adjointness of $\mathcal{D},\mathcal{E}$ and for (i.c) we further need to notice that $\mathsf{pr}\,\mathbf{v}_{\mathcal{D}\cdot}(\mathcal{E})\Delta = (\mathsf{pr}\,\mathbf{v}_{\cdot}(\mathcal{E})\Delta)\mathcal{D}$ and (12) for functional bi-vectors i.e. skew-adjoint operators $\mathcal{K} : \mathcal{F}^1 \to \mathcal{V}^1$. □

Definition 5 (Poisson bracket). *Let* $\mathcal{D} : \mathcal{F}^1 \to \mathcal{V}^1$ *be a differential operator. The Poisson bracket of two functionals* L, P *is defined by*

$$
\{L,P\} = \mathsf{E}(L) \cdot \mathcal{D}\mathsf{E}(P), \tag{16}
$$

which is again a functional.

Definition 6 (Hamiltonian structure). *A differential operator $\mathcal{D} : \mathcal{F}^1 \to \mathcal{V}^1$ is called Hamiltonian if its Poisson bracket (16) is skew-symmetric*

$$\{L, P\} = -\{P, L\}, \tag{17}$$

and satisfies the Jacobi identity

$$\{\{L, P\}, R\} + \{\{R, L\}, P\} + \{\{P, R\}, L\} = 0, \tag{18}$$

for all functionals L, P, R. These are identities between functionals.

Proposition 3. *A differential operator $\mathcal{D}$ is Hamiltonian, if and only if $\mathcal{D}$ is a 2-vector satisfying $[\mathcal{D}, \mathcal{D}] = 0$.*

Proof. First we note that if we replace $\mathcal{E}$ by $\mathcal{D}$ in the right hand side of (15), then, up to a factor, we obtain (7.11) in [Olv]. The rest is done by [Olv] Propositions 7.3, 7.4. □

Definition 7 (Hamiltonian equations). *Let $K \in \mathcal{V}^1$ and $u_t = K$ a system of evolution equations. We say the evolution equation is Hamiltonian, if there exists a Hamiltonian structure $\mathcal{D}$ and a functional H, such that*

$$K = \mathcal{D}\mathsf{E}(H). \tag{19}$$

Definition 8 (Bi-Hamiltonian structure). *Let $u_t = \mathcal{E}\mathsf{E}(H_0) = \mathcal{D}\mathsf{E}(H_1)$ be a system with two Hamiltonian structures. The system is called bi-Hamiltonian, if $[\mathcal{D}, \mathcal{E}] = 0$.*

Remark 3. Bi-Hamiltonian systems with a nondegenerate $\mathcal{D}$ possess a recursion operator $\mathcal{R} = \mathcal{E}\mathcal{D}^{-1}$ generating an infinite family of Hamiltonian symmetries, which by Noether's theorem give rise to an infinite family of conservation laws. This is typical for an integrable system. Details are found in [Olv] and [Bar].

Example 1. The KdV equation $u_t = u_{xxx} + uu_x$ has a bi-Hamiltonian structure with $\mathcal{D} = D_x$, $H_1 = \frac{1}{6}u^3 - \frac{1}{2}u_x^2$, $\mathcal{E} = D_{xxx} + \frac{2}{3}uD_x + \frac{1}{3}u_x$ and $H_0 = \frac{1}{2}u^2$.

3 Example

In this section we use `jets` to compute the set of all Hamiltonian structures of order three and jets up to order one, compatible with $\mathcal{D} = D_x$, obtaining candidates for nonlinear integrable systems.

```
> restart;
```

Loading the package:

```
> read`maple/lib/desolv`: with(jets):
```

Defining the list of independent and dependent variables ($p = q = 1$):

```
> ivar:=[x]; dvar:=[u]; var:=op(alljets(1,ivar,dvar));
```

$$ivar := [x]$$

$$dvar := [u]$$

$$var := x, u, u_x$$

Defining the operator $\mathcal{D}$:

```
> DD:=[[1,[x]]];
```

$$DD := [[1, [x]]]$$

Defining the general operator $\mathcal{E} : \mathcal{F}^1 \to \mathcal{V}^1$ of order at most 3, depending on jet variables of order at most 1, on which we impose several conditions:

```
> SMP:=[[Qxxx(var),[x,x,x]],[Qxx(var),[x,x]],
[Qx(var),[x]],[Q(var),[]]];
```

$$SMP := [[Qxxx(x, u, u_x), [x, x, x]],\\ [Qxx(x, u, u_x), [x, x]], [Qx(x, u, u_x), [x]], [Q(x, u, u_x), [\,]]]$$

The first condition is the compatibility of $\mathcal{D}$ and $\mathcal{E}$, i.e. they must commute (Definition 8). This is a linear condition for $\mathcal{E}$:

```
> BRA:=nsbra3(DD,SMP,ivar,dvar):
```

Here we extract the linear conditions, for the resulting functional 3-vector to vanish. We use the fact that $[\mathcal{D}, \mathcal{E}](T)$ is a total differential operator in $T = (T^{u^1}, \ldots, T^{u^q})$:

```
> CND1:=getcond(map(a->a[1],BRA),map(a->a[1],SMP),
ivar,[op(dvar),T.(op(dvar))]):
```

'jsolve' is a wrapper function which uses the package `desolv` to solve the given system of linear PDEs:

```
> jsolve(CND1): RES1:=subs(CND1[4],%):
```

The third list is the general solution, and the fourth list is the list of all functions and constants appearing in the general solution. The first empty list means that `desolv` succeeded to completely solve the system:

```
> RES1;
```

$$[[\,], [\,], [Qxxx(x, u, u_x) = F\_3(x), Qxx(x, u, u_x) = F\_4(x),\\ Qx(x, u, u_x) = F\_5(x, u),\\ Q(x, u, u_x) = \frac{1}{2}\,(\frac{\partial}{\partial x}\,F\_5(x, u)) + \frac{1}{4}\,F\_8(x) + \frac{1}{2}\,u_x\,(\frac{\partial}{\partial u}\,F\_5(x, u))],\\ [F\_5(x, u), F\_4(x), F\_3(x), F\_8(x)]]$$

Define the intermediate $\mathcal{E}$, i.e. the general $\mathcal{E}$ that satisfies the linear condition $[\mathcal{D},\mathcal{E}]=0$:

```
> SMP1:=convert(subs(RES1[3],SMP),D);
```

$$SMP1 := [[F\_3(x),[x,x,x]],\ [F\_4(x),[x,x]],[F\_5(x,u),[x]], [\frac{1}{2}D_1(F\_5)(x,u)+\frac{1}{4}F\_8(x)+\frac{1}{2}u_x\,D_2(F\_5)(x,u),[\,]]]$$

$\mathcal{E}$ is skew-adjoint. This condition also produces a linear system of PDEs:

```
> sadj(SMP1,ivar,dvar):
CND2:=getcond(map(a->a[1],%),RES1[4],ivar,dvar);
```

$$CND2 := [F\_4(x)-\frac{3}{2}(\frac{\partial}{\partial x}F\_3(x)),-\frac{3}{2}(\frac{\partial^2}{\partial x^2}F\_3(x))+(\frac{\partial}{\partial x}F\_4(x)), \frac{1}{4}F\_8(x)-\frac{1}{2}(D^{(3)})(F\_3)(x)+\frac{1}{2}(D^{(2)})(F\_4)(x)], [F\_5(x,u),F\_4(x),F\_3(x),F\_8(x)],[x,u],[x=x,u=u]$$

Find the solution and redefine the intermediate $\mathcal{E}$:

```
> RES2:=jsolve(CND2):
SMP2:=convert(esubs(RES2[3],SMP1),D);
```

$$SMP2 := [[F\_3(x),[x,x,x]],\ [\frac{3}{2}D(F\_3)(x),\ [x,x]],[F\_5(x,u),[x]], [\frac{1}{2}D_1(F\_5)(x,u)-\frac{1}{4}(D^{(3)})(F\_3)(x)+\frac{1}{2}u_x\,D_2(F\_5)(x,u),[\,]]]$$

For $\mathcal{E}$ to be Hamiltonian, $\mathcal{E}$ must commute with itself (Definition 3):

```
> nsbra3(SMP2,SMP2,ivar,dvar):
CND3:=getcond(map(a->a[1],%),RES2[4],
ivar,[op(dvar),T.(op(dvar))]):
```

The resulting conditions form a nonlinear system of PDEs:

```
> CND3;
```

$$[2\,\%1\,F\_3(x),2\,\%4\,F\_3(x)-\%5,-\frac{3}{2}\%5+3\,\%4\,F\_3(x),$$
$$-\frac{3}{2}D_2(F\_5)(x,u)\,(D^{(2)})(F\_3)(x)+\frac{3}{2}\%4\,D(F\_3)(x)+3\,F\_3(x)\,\%6,$$
$$3\,\%1\,F\_3(x),3\,F\_3(x)\,\%3,6\,F\_3(x)\,\%2+\frac{3}{2}\%1\,D(F\_3)(x),$$
$$3\,\%1\,F\_3(x),\frac{3}{2}D(F\_3)(x)\,\%3+3\,F\_3(x)\,D_{1,2,2,2}(F\_5)(x,u),$$
$$\frac{3}{2}\%1\,D(F\_3)(x)+3\,F\_3(x)\,\%2,-3\,\%1\,F\_3(x),$$
$$3\,D(F\_3)(x)\,\%2+3\,F\_3(x)\,D_{1,1,2,2}(F\_5)(x,u),\%1\,F\_3(x),$$

$$3\,F\_3(x)\,\%3, F\_3(x)\,D_{2,2,2,2}(F\_5)(x,u), F\_3(x)\,D_{1,1,1,2}(F\_5)(x,u)$$
$$+\frac{3}{2}\,D(F\_3)(x)\,\%6-\frac{1}{2}\,D_2(F\_5)(x,u)\,(D^{(3)})(F\_3)(x),$$
$$\frac{3}{2}\,\%5-3\,\%4\,F\_3(x), -3\,F\_3(x)\,\%3, -\%1\,F\_3(x),$$
$$-F\_3(x)\,D_{2,2,2,2}(F\_5)(x,u), -3\,\%1\,F\_3(x),$$
$$-3\,F\_3(x)\,\%2-\frac{3}{2}\,\%1\,D(F\_3)(x), -3\,F\_3(x)\,\%3,$$
$$-3\,D(F\_3)(x)\,\%2-3\,F\_3(x)\,D_{1,1,2,2}(F\_5)(x,u),$$
$$\frac{1}{2}\,D_2(F\_5)(x,u)\,(D^{(3)})(F\_3)(x)-F\_3(x)\,D_{1,1,1,2}(F\_5)(x,u)-\frac{3}{2}\,D(F\_3)(x)\,\%6,$$
$$-\frac{3}{2}\,\%4\,D(F\_3)(x)-3\,F\_3(x)\,\%6+\frac{3}{2}\,D_2(F\_5)(x,u)\,(D^{(2)})(F\_3)(x),$$
$$\%5-2\,\%4\,F\_3(x), -\frac{3}{2}\,D(F\_3)(x)\,\%3-3\,F\_3(x)\,D_{1,2,2,2}(F\_5)(x,u),$$
$$-6\,F\_3(x)\,\%2-\frac{3}{2}\,\%1\,D(F\_3)(x), -2\,\%1\,F\_3(x)],$$
$$[F\_3(x), F\_5(x,u)], [u,x], [x=x, u=u, Tu=Tu, u[x]=u_x,$$
$$Tu[x,x,x]=Tu_{x,x,x}, Tu[x,x]=Tu_{x,x}, u[x,x]=u_{x,x},\ u[x,x,x]=u_{x,x,x},$$
$$Tu[x]=Tu_x]$$
$$\%1 := D_{2,2}(F\_5)(x,u)$$
$$\%2 := D_{1,2,2}(F\_5)(x,u)$$
$$\%3 := D_{2,2,2}(F\_5)(x,u)$$
$$\%4 := D_{1,2}(F\_5)(x,u)$$
$$\%5 := D_2(F\_5)(x,u)\,D(F\_3)(x)$$
$$\%6 := D_{1,1,2}(F\_5)(x,u)$$

Because we assume $F\_3(x) \neq 0$, the first equation in `CND3` yields:

```
> SUBS:=SMP2[3,1]=G_1(x)*u+G_2(x);
```

$$SUBS := F\_5(x,u) = G\_1(x)\,u + G\_2(x)$$

Reinserting the new $F\_5(x)$ in `CND3`:

```
> esubs(SUBS,CND3[1]);
```

$$[0, 2\,D(G\_1)(x)\,F\_3(x)-G\_1(x)\,D(F\_3)(x),$$
$$-\frac{3}{2}\,G\_1(x)\,D(F\_3)(x)+3\,D(G\_1)(x)\,F\_3(x),$$
$$-\frac{3}{2}\,G\_1(x)\,(D^{(2)})(F\_3)(x)+\frac{3}{2}\,D(G\_1)(x)\,D(F\_3)(x)$$
$$+3\,F\_3(x)\,(D^{(2)})(G\_1)(x), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$F\_3(x)\,(D^{(3)})(G\_1)(x)+\frac{3}{2}\,D(F\_3)(x)\,(D^{(2)})(G\_1)(x)$$

$$
\begin{aligned}
&-\frac{1}{2}\,G\_1(x)\,(D^{(3)})(F\_3)(x),\\
&\frac{3}{2}\,G\_1(x)\,D(F\_3)(x)-3\,D(G\_1)(x)\,F\_3(x),0,0,0,0,0,0,0,\\
&\frac{1}{2}\,G\_1(x)\,(D^{(3)})(F\_3)(x)-F\_3(x)\,(D^{(3)})(G\_1)(x)\\
&-\frac{3}{2}\,D(F\_3)(x)\,(D^{(2)})(G\_1)(x),-\frac{3}{2}\,D(G\_1)(x)\,D(F\_3)(x)\\
&-3\,F\_3(x)\,(D^{(2)})(G\_1)(x)+\frac{3}{2}\,G\_1(x)\,(D^{(2)})(F\_3)(x),\\
&G\_1(x)\,D(F\_3)(x)-2\,D(G\_1)(x)\,F\_3(x),0,0,0]
\end{aligned}
$$

The first nonzero equation:

```
> eqn:=2*D(G_1)(x)*F_3(x)-G_1(x)*D(F_3)(x);
```

$$eqn := 2\,D(G\_1)(x)\,F\_3(x)-G\_1(x)\,D(F\_3)(x)$$

Solve with MAPLE's internal 'dsolve' command:

```
> sol:=dsolve(eqn,F_3(x));
```

$$sol := F\_3(x) = \_C1\,G\_1(x)^2$$

The solution of the first equation satisfies all other equations!

```
> simplify(esubs([SUBS,sol],CND3[1]));
```

$$[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

Define the final $\mathcal{E}$, i.e. the most general Hamiltonian operator of order at most 3, depending on jet variables of order at most 1:

```
> symp:=esubs([SUBS,sol],SMP2);
```

$$
\begin{aligned}
&symp := [[\_C1\,G\_1(x)^2,\ [x,x,x]],[3\,\_C1\,G\_1(x)\,D(G\_1)(x),[x,x]],\\
&[G\_1(x)\,u+G\_2(x),[x]],[\frac{1}{2}\,D(G\_1)(x)\,u+\frac{1}{2}\,D(G\_2)(x)\\
&-\frac{3}{2}\,\_C1\,D(G\_1)(x)\,(D^{(2)})(G\_1)(x)-\frac{1}{2}\,\_C1\,G\_1(x)\,(D^{(3)})(G\_1)(x)\\
&+\frac{1}{2}\,u_x\,G\_1(x),[\,]]]
\end{aligned}
$$

Check skew-adjointness:

```
> sadj(symp,ivar,dvar);
```

$$0$$

Check compatibility:

```
> nsbra3(DD,symp,ivar,dvar);
```

$$0$$

Check the Hamiltonian condition:

```
> nsbra3(symp,symp,ivar,dvar);
```

$$0$$

Example 1 ([GDo1]):

```
> EX1:=gcollect(
esubs([_C1=0,G_1(x)=2*C[2],G_2(x)=2*C[1]],symp),ivar);
```

$$EX1 := [[2\,C_2\,u + 2\,C_1, [x]], [u_x\,C_2, [\,]]]$$

```
> CEX1:=gcollect(
esubs([_C1=1/(4*D[2]^2),G_1(x)=2*D[2]],symp),ivar);
```

$$CEX1 := [[1, [x,x,x]], [2\,D_2\,u + G\_2(x), [x]], [\frac{1}{2}\,D(G\_2)(x) + u_x\,D_2, [\,]]]$$

The last two operators form a Hamiltonian pair (C_1,C_2,D_2 constants):

```
> nsbra3(EX1,CEX1,ivar,dvar);
```

$$0$$

Example 2 (KdV):

```
> gcollect(
esubs([_C1=9/4,G_1(x)=2/3,G_2(x)=f(x)],symp),ivar);
```

$$[[1, [x,x,x]], [\frac{2}{3}\,u + f(x), [x]], [\frac{1}{2}\,D(f)(x) + \frac{1}{3}\,u_x, [\,]]]$$

For $f(x) = 0$ we obtain the KdV-operator:

```
> KdV:=[[1,[x,x,x]],[2/3*u,[x]],[1/3*u[x],[]]];
```

$$KdV := [[1, [x,x,x]], [\frac{2}{3}\,u, [x]], [\frac{1}{3}\,u_x, [\,]]]$$

If we choose $G\_1(x) = D_2 = const$ then $\mathcal{E}$ commutes even with the more general operator $(2C_2u + 2C_1)D_x + C_2u_x$:

```
> nsbra3(EX1,esubs(G_1(x)=D[2],symp),ivar,dvar);
```

$$0$$

References

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[GDo1] I. M. Gel'fand and I. Ya. Dorfman. *Hamiltonian operators and algebraic structures related to them.* Func. Anal. Appl. 13 (1979), 248-262.

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