Proof (of Lemma 7): (a) Observe, that $K \subseteq A_\varepsilon$, and $A_\varepsilon \cdot A_\varepsilon \subseteq A_{\varepsilon \circ \varepsilon} = A_\varepsilon$

(b) $\forall \gamma \in \Gamma : A_\varepsilon \cdot A_\gamma \subseteq A_{\varepsilon \circ \gamma} = A_\gamma$; same for $A_\gamma \cdot A_\varepsilon \subseteq A_\gamma$

$(r_\varepsilon \cdot a_\gamma) \cdot r_\varepsilon' = r_\varepsilon (a_\gamma \cdot r_\varepsilon')$ assoc. multiplication of A $\forall r_\varepsilon, r_\varepsilon' \in A_\varepsilon, a_\gamma \in A_\gamma$ □

Definition 9: $\forall a \in A \setminus \{0\}$ a Γ-graded algebra, $a = \underbrace{a_\alpha + \ldots + a_\omega}_{\text{finitely many } a_\gamma \in A_\gamma}$

(if Γ is an ordered monoid, then $\alpha > \ldots > \omega$,

a_γ "homogeneous of degree γ" or "graded" of degree γ.)

Multiplication: $\underset{(a_\alpha + \ldots + a_\omega)}{\left(\sum_{\gamma \in I} a_\gamma\right)} \cdot \underset{(b_{\alpha'} + \ldots + b_{\omega'})}{\left(\sum_{\delta \in J} b_\delta\right)} = \sum_{\kappa \in K} \underbrace{\left(\sum_{\gamma \circ \delta = \kappa} a_\gamma b_\delta\right)}_{A_\kappa} = (a_\alpha b_{\alpha'}) + \ldots + (a_\omega b_{\omega'})$

$= c_{\alpha''} + \ldots + c_{\omega''}$.

Homework: (1) $A = K[X,Y]$, $\underbrace{\deg(X) = 3, \deg(Y) = 5}_{\mathbb{N}_0\text{-grading}}$: describe $A_0, A_\gamma, \gamma \in \mathbb{N}$

(2) $A = K[X,Y]$, $\deg(X) = -1$, $\deg(Y) = 1$: describe A_0, A_{+k}, A_{-k}. $k \in \mathbb{Z}$

(3) over the Weyl algebra over an abstract K (or rather char$K = 0$)

$\partial \cdot x^n = x^n \cdot \partial + n \cdot x^{n-1}$, more generally

$\forall f \in K[t]$: $\partial \cdot f(x) = f(x)\partial + \frac{\partial f}{\partial x}(x)$

- $\partial^m x = x \partial^m + m \partial^{m-1}$ $\forall m, n \in \mathbb{N}$
- prove: $\partial^m x^n = \sum_{k=0}^{\min\{m,n\}} \frac{m!\,n!}{k!} \frac{x^{n-k} \cdot \partial^{m-k}}{(m-k)!\,(n-k)!}$
- $\theta = x \cdot \partial$, check that $x^n \cdot f(\theta) = f(\theta - n) \cdot x^n$

$\partial^n \cdot f(\theta) = f(\theta + n) \cdot \partial^n$ (*)

$\partial^m \cdot x^m = \prod_{i=1}^{m} (\theta + i)$, $x^m \partial^m = \prod_{i=0}^{m-1} (\theta - i)$

(4) compare (2) with considerations about $\mathbb{Z}$-grading ($\deg \partial = 1$, $\deg x = -1$).

$\boxed{17.04.19}$

Solution: (1). $A_0 = K$, $A_1 = 0 = A_2$, $A_3 = K \cdot X$, $A_4 = 0$, $A_5 = K \cdot y$, $A_{15} = K \cdot x^5 \oplus K \cdot y^3$.

$\mathbb{N} \cdot \langle 3,5 \rangle = \{3,5,6,8,9,\ldots\}$

(2) $A_0 = K[X \cdot Y] \ni (xy)^i = x^i \cdot y^i$

$A = {}_K\{X^a y^b \mid (a,b) \in \mathbb{N}_0^2\}$, $A_i = \begin{cases} K[XY] \cdot y^i, & i \geq 0 \\ K[XY] \cdot x^i, & i < 0 \end{cases}$

$-a + b = i \neq 0$

$\Leftrightarrow b = a + i$

$\left(f \in K[X,Y],\ f = \underbrace{f_7(X \cdot Y)}_{\in K[X,Y]} \cdot Y^7 + \ldots + f_0(X \cdot Y) + \ldots + f_{-42}(X \cdot Y) X^{42}\right)$

Remark: Hilbert series of A if A_0 is a field:

$$\sum_{i=0}^{\infty} (\dim_{A_0} A_i) t^i \in \mathbb{Z}[[t]]$$

(3) 1st Weyl algebra: $A = K\langle x, \partial \mid \partial x = x\partial + 1\rangle$

<u>Lemma:</u> a) $\deg(x) = -1$, $\deg(\partial) = 1$ provides a $\mathbb{Z}$-grading on A

b) $A_0 = K[\theta]$, $\theta = x \cdot \partial$ the Euler operator $(x\partial)\cdot(x^m) = m x^m$

$$A_z = \begin{cases} K[\theta]\, x^{|z|}, & z < 0 \\ K[\theta]\, \partial^z, & z \geq 0 \end{cases}$$

<u>proof:</u> K-basis of A is $\{x^a \partial^b \mid (a,b) \in \mathbb{N}_0^2\}$

$x^{a'}\underbrace{(x^{a''}\partial^{b''})}_{\in K[\theta]}$ $\quad \underbrace{(x^{a'}\partial^{b'})}_{\in K[\theta]}\partial^{b''}$.

(*) for $n=1$, $f=t$: $x \cdot \theta = (\theta - 1)\cdot x = \theta x - x$,

$\partial \theta = \theta \cdot \partial + \partial$

$\tau: K\langle k, S_k \mid S_k \cdot k = (k+1) S_k = k S_k + S_k \rangle \longrightarrow K\langle x, \partial \mid \partial x = x\partial + 1\rangle$,

$k \mapsto x\partial$, $S_k \mapsto \partial$

<u>Homework:</u> Compute $\ker(\tau)$, $\operatorname{im}(\tau)$.

Factorization in Weyl + algebras (L., Heinle, Giesbrecht, Bell)

Observe: $\partial \cdot x = (x\partial + 1)\cdot 1 = \theta + 1$

① The only monic elements in $K[\theta]$ which are irred., but factorize in A_1 (Weyl algebra) are θ, $\theta + 1$.

② Any factorization of a graded element consists of graded elements.

<u>proof:</u> Let $(\Gamma, 0, \varepsilon, <)$ be an ordered monoid. Let A be a domain.

$g \in A_\gamma$, $\gamma \in \Gamma$, $g = p \cdot q$

$\alpha(p) :=$ highest Γ-part, $\omega(p) :=$ lowest Γ-part of p

$\alpha(pq) = \alpha(p) + \alpha(q)$, same for ω

$\alpha, \omega: A \to \Gamma$

h is graded $\Leftrightarrow \alpha(h) = \omega(h) = h_{\alpha(h)} = h_{\omega(h)}$.

$\alpha(g) = \alpha(p) + \alpha(q) \geq \omega(p) + \omega(q) = \omega(pq) = \omega(g) = \alpha(g) \Rightarrow \alpha(p) = \omega(p)$,

same for q. □

③ Graded-driven factorization under mild assumptions (A_z is finitely generated A_0-bimodule)

$h = \sum_{\gamma \in I} h_\gamma = h_{\alpha(h)} + \ldots + h_{\omega(h)} \overset{\text{suppose}}{=} p \cdot q = (p_{\alpha(p)} + \ldots + p_{\omega(p)}) \cdot (q_{\alpha(q)} + \ldots + q_{\omega(q)})$

$h_{\alpha(h)} = p_{\alpha(p)} \cdot q_{\alpha(q)}$, $h_{\omega(h)} = p_{\omega(p)} \cdot q_{\omega(q)}$, since $\alpha(h) = \alpha(p) + \alpha(q)$

Classical Ansatz:

we use coefficients at monomials as variables

$x^2+3x+42=(x-a)(x-b) = x^2-(a+b)x+ab \quad \Rightarrow \quad \begin{matrix} a+b=-3 \\ a\cdot b = 42 \end{matrix}$

<u>Definition</u> ⓐ Let A be a Γ-graded algebra and M be a left A-module ($\forall m\in M, a\in A: a\cdot m\in M$). M is called a <u>Γ-graded left A-module</u>, if

(i) $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$

(ii) $A_\alpha M_\gamma \subseteq M_{\alpha\circ\gamma} \quad \forall \alpha,\gamma\in\Gamma$

ⓑ Let A, B be two Γ-graded K-algebras, $\varphi: A=\bigoplus_{\gamma\in\Gamma} A_\gamma \longrightarrow B=\bigoplus_{\gamma\in\Gamma} B_\gamma$ a homomorphism of K-algebras. φ is called a <u>Γ-graded homomorphism</u>, if $\forall\gamma\in\Gamma: \varphi(A_\gamma)\subseteq B_\gamma$, [since: $\varphi(A_\alpha A_\gamma)=\varphi(A_\alpha)\cdot\varphi(A_\gamma)\subseteq\varphi(A_{\alpha\circ\gamma})$, $\varphi(A_0)\subseteq B_0$]

ⓒ Let M, N be two Γ-graded A-moduls, $\psi: M=\bigoplus M_\gamma \longrightarrow N=\bigoplus N_\gamma$ a homomorphism of A-moduls. ψ is called a <u>Γ-graded homomorphism with shift $\delta\in\Gamma$</u>, if $\psi(M_\gamma)\subseteq N_{\gamma\circ\delta}\ \forall\gamma\in\Gamma$.

<u>Example</u>: $I(f(x)) = \int_0^x f(t)\,dt$

algebra generated by x, ∂, I : $\begin{cases} I\cdot x = x\cdot I - I^2 \\ \partial\cdot x = x\partial + 1 \\ \partial\cdot I = 1 \end{cases}$

$(I\partial)(f(x)) = \int_0^x \frac{\partial f}{\partial t}(t)\,dt = f(x)-f(0)$

$(1-I\partial)(f) = f(0)$ Evaluation

$(1-I\partial)^2 = (1-I\partial)$

Homework: $\mathbb{Z}$-grading?