Let $\mathcal{F}^*_\gamma M := \sum_{\beta<\gamma} \mathcal{F}_\beta$, then $\quad gr^{\mathcal{F}} M := \bigoplus \mathcal{F}_\gamma M / \mathcal{F}^*_\gamma M \quad$ is an assoc. $\Gamma$-graded left-module over $gr^{\mathcal{F}} A$.

$$G_\alpha A \times G_\beta M \longrightarrow G_{\alpha\circ\beta} M \;, \; (a + \mathcal{F}^*_\alpha A, m + \mathcal{F}^*_\beta M) \mapsto a \cdot m + \mathcal{F}^*_{\alpha\circ\beta} M .$$

Note: For a submodule $N \subseteq M$, there is an induced $\Gamma$-filtration

$$\{ \tilde{\mathcal{F}}_\gamma N := \mathcal{F}_\gamma M \cap N \mid \gamma \in \Gamma \} \;, \quad \{ \mathcal{F}_\gamma (M/N) := \mathcal{F}_\gamma M + N / N \mid \gamma \in \Gamma \}.$$

K-algebras $\quad A \xrightarrow{\varphi} B$

$\Gamma$-filtrations $\quad \mathcal{F}A \quad \tilde{\mathcal{F}}B$

$\varphi$ is $(\Gamma, \mathcal{F}, \tilde{\mathcal{F}})$-filtered $\iff \varphi(\mathcal{F}_\gamma A) \subseteq \tilde{\mathcal{F}}_\gamma B$

$\Gamma$-filtered modules $\quad M^{\mathcal{F}} \xrightarrow{\varphi} N^{\tilde{\mathcal{F}}}$

$\varphi$ is $\Gamma$-filtered, if $\varphi(\mathcal{F}_\gamma M) \subseteq \tilde{\mathcal{F}}_\gamma N \ \forall \gamma \in \Gamma$

$\varphi$ is strict, if $\varphi(\mathcal{F}_\gamma M) = \varphi(M) \cap \tilde{\mathcal{F}}_\gamma N \ \forall \gamma \in \Gamma$.

$\boxed{8.05.2019}$

Remark 3: Let $N \subseteq M$ a submodule, $\mathcal{F}M$ be a filtration on $M$, define $\mathcal{F}N := \{ \mathcal{F}_\gamma N := N \cap \mathcal{F}_\gamma M \mid \gamma \in \Gamma \}$ to be the induced filtration
$\mathcal{F}(M/N) := \{ \mathcal{F}_\gamma (M/N) := (\mathcal{F}_\gamma M + N)/N \mid \gamma \in \Gamma \}$.
Then $i : N \hookrightarrow M$, $\pi : M \longrightarrow M/N$ are strict $\Gamma$-filtered homomorphisms.

Proposition 4: Let $\varphi : M \to N$ be a $\Gamma$-filtered homomorphism. Then
$Im(\varphi) \subseteq N$ is $\Gamma$-filtered with $\mathcal{F}(Im\,\varphi) := \{ \varphi(\mathcal{F}_\gamma M) \}$,
$Ker(\varphi) \subseteq M$ is $\Gamma$-filtered with $\mathcal{F}(Ker\,\varphi) := \{ \mathcal{F}_\gamma M \cap Ker\,\varphi \}$.
There is a $\Gamma$-graded homomorphism of $G^{\mathcal{F}}A$-modules

$$G^{\mathcal{F}}(\varphi) : G^{\mathcal{F}} M = \bigoplus \mathcal{F}_\gamma M / \mathcal{F}_{\gamma^*} M \longrightarrow G^{\mathcal{F}} N = \bigoplus \tilde{\mathcal{F}}_\gamma N / \tilde{\mathcal{F}}_{\gamma^*} N \,,$$

$$\sum [m_\gamma] \longmapsto \sum [\varphi(m_\gamma)]$$

$gr^{\mathcal{F}}$: $\{\Gamma\text{-filtered rings}\} \longrightarrow \{\Gamma\text{-graded rings}\}$

$\{\Gamma\text{-filterd modules}/\text{a } \Gamma\text{-filtered ring}\} \longrightarrow \{\Gamma\text{-graded modules}/\text{a } \Gamma\text{-graded ring}\}$

Assume: $\Gamma$ is a well-ordered monoid and $e$ is the smallest element.

$m \in M \setminus 0$, the degree of $m$ (wrt. $\Gamma, \mathcal{F}, ..$) is $d(m) := \min\{\gamma \mid m \in \mathcal{F}_\gamma M\} \in \Gamma$,

then the principal symbol of $m$ $\sigma(m)$ is the nonzero graded element of degree $d(m)$ in $\mathcal{G}^{\mathcal{F}}_{d(m)} M = \mathcal{F}_{d(m)} M / \mathcal{F}_{d(m)^*} M$.

Note: $\forall a \in A, m \in M$ either $\sigma(a)\sigma(m) = \sigma(am)$ or $\sigma(a)\sigma(m) = 0$.

Theorem 5: Let $A$ be a $\Gamma$-filtered algebra (with $\mathcal{F}A$).

(a) If $\{a_\gamma \mid \gamma \in \Gamma\} \subseteq A$ is such, that $\{\sigma(a_\gamma) \mid \gamma \in \Gamma\}$ is a $K$-basis of $gr^{\mathcal{F}} A$, then $\{a_\gamma \mid \gamma \in \Gamma\}$ is a $K$-basis of $A$.

(b) If $gr^{\mathcal{F}} A$ is a domain, so is $A$.

proof: (b) $a, b \in A \setminus 0$, $\alpha := \deg(a)$, $\beta := \deg(b)$, then $\sigma(a), \sigma(b) \neq 0$

$\Rightarrow \sigma(a) \cdot \sigma(b) \neq 0$ in $gr^{\mathcal{F}} A \overset{[Note]}{\Rightarrow} \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) \neq 0 \underset{\sigma(a) \neq 0 \Leftrightarrow a \neq 0}{\Rightarrow} a \cdot b \neq 0$.

(a) • show that $\{a_\gamma\}$ is lin. indep. Suppose $\exists \lambda_\delta \in K$ such that

$b = \sum \lambda_\delta a_\delta = 0 \Rightarrow \exists$ subset $\Lambda$ of indices, s.t. $\lambda_\omega \neq 0, \omega \in \Lambda$

and let $\gamma :=$ highest degree among the corresponding $a$'s.

Then the image of $b$ in $\mathcal{F}_\gamma A / \mathcal{F}_{\gamma^*} A \subseteq gr^{\mathcal{F}} A$ is

$0 = \sum_{\delta \in \Lambda} \lambda_\delta \sigma(a_\delta)$ and since $\{\sigma(a_\gamma)\}$ is lin. indep. $\Rightarrow \Lambda = \emptyset$

• The generating property: Let $a \in A \setminus 0$, $\gamma := \min\{\alpha \mid a \in \mathcal{F}_\alpha A\} = \deg(a)$.

Since $\sigma(a) = \sum_{i \in I} \lambda_i \sigma(a_i) \in \mathcal{F}_\gamma A / \mathcal{F}_{\gamma^*} A$ ; $\lambda_i \in K \setminus 0$, $\deg(\sigma(a_i)) = \gamma \quad \forall i \in I$

$a' := a - \sum_{i \in I} \lambda_i a_i \in \mathcal{F}_{\gamma^*} A$

$\sigma(a) = a + \mathcal{F}_{\gamma^*} A = \underbrace{a_\gamma}_{\text{"part" of } a \text{ of degree } \gamma} + \mathcal{F}_{\gamma^*} A$

If $a' = 0$, we're done.

Otherwise $\deg(a') < \gamma$ and proceeding like the above will terminate in a finite number of steps. (Artinian) $\square$

Theorem 6: Let $L \xrightarrow{\varphi} M \xrightarrow{\psi} N$ (*) be a complex (i.e. $\psi \circ \varphi = 0$) of $\Gamma$-filtered modules and morphisms. Then the following are equivalent:

(i) The sequence is exakt ($\ker \psi = \operatorname{im} \varphi$) and $\varphi, \psi$ are strict.

(ii) The associated sequence of $gr^{\mathcal{F}} A$-modules with ass. graded $gr^{\mathcal{F}} A$-module-hom.'s $grL \xrightarrow{gr\varphi} grM \xrightarrow{gr\psi} grN$ is exakt. (gr*)

<u>Proposition 7:</u> (i) If $M$ is fin. gen. over $A$, then there exists a $\Gamma$-filtered $\tilde{\mathcal{F}}M$, such that $gr^{\tilde{\mathcal{F}}}M$ is a fin. gen. $gr^{\mathcal{F}}A$-module.

(ii) Suppose that $M$ is $\Gamma$-filtered via $\tilde{\mathcal{F}}$. If $gr^{\tilde{\mathcal{F}}}M = \sum_{i\in I} gr^{\mathcal{F}}A\cdot\sigma(\xi_i)$, where $\xi_i\in M$, $\deg(\xi_i)=\gamma_i\in\Gamma$, then $M=\sum_{i\in I}A\cdot\xi_i$, with $\boxed{\tilde{\mathcal{F}}_\gamma M := \sum_{i\in I}\Big(\sum_{\delta_i+\gamma_i\le\gamma}\mathcal{F}_{\delta_i}A\Big)\cdot\xi_i}$.
In particular, if $gr^{\tilde{\mathcal{F}}}M$ is fin. gen., then so is $M$.

<u>proof:</u> (i) WLOG $\{g_1,\dots,g_n\}$ is a minimal gen. set of $M$.

$\forall m\in M$: $m=\sum_{i=1}^n a_i g_i$. Let $\delta_i\in\Gamma$ be such, that $a_i\in \mathcal{F}_{\delta_i}A/\mathcal{F}_{\delta_i^*}A$, then

$\forall\gamma\in\Gamma$: $\tilde{\mathcal{F}}_\gamma M := \sum_{i=1}^n\Big(\sum_{\delta_i+\gamma_i\le\gamma}\mathcal{F}_{\delta_i}A\Big)\cdot g_i$.

Define a $\Gamma$-filtration on $M$, where $\gamma_i := \deg(g_i)$. Then

$$(gr^{\tilde{\mathcal{F}}}M)_\gamma = \tilde{\mathcal{F}}_\gamma M/\tilde{\mathcal{F}}_{\gamma^*}M = \sum_{i=1}^n\sum_{\delta_i+\gamma_i=\gamma}(gr^{\tilde{\mathcal{F}}}A)_{\delta_i}\cdot\sigma(g_i)$$

$$gr^{\tilde{\mathcal{F}}}M = \sum_{i=1}^n gr^{\mathcal{F}}A\cdot\sigma(g_i)$$

□

(ii) Homework.

<u>proof of theorem 6:</u> (ii)$\Rightarrow$(i): $\Gamma$, $\mathcal{F}$ are fixed.

"exactness": Let $(gr\,*)$ be exact. For $m\in\mathcal{F}_\gamma M\setminus\mathcal{F}_{\gamma^*}M$ with $\psi(m)=0$, $(gr\,\psi)([m])=0$. Since $\deg(m)=\gamma$, $[m]\in(gr\,\psi)(M)_\gamma$, thus $[m]=(gr\,\varphi)(\ell)$ for some $\ell\in\mathcal{F}_\gamma L\setminus\mathcal{F}_{\gamma^*}L$ and $[m]=[\varphi(\ell)]$.

$m':=m-\varphi(\ell)\in\mathcal{F}_{\gamma'}M$, $\gamma'<\gamma$, then $\psi(m')=\psi(m-\varphi(\ell))=0$.

Iterating: $m'':=m'-\varphi(\ell')$ ...

$\Rightarrow$ After fin. many steps $m'^{\cdots\prime}=0$

$\Rightarrow$ $m=\sum\varphi(\ell^{(i)})=\varphi(\sum\ell^{(i)})\in\operatorname{Im}(\varphi)$

$\Rightarrow$ $m\in\operatorname{Im}\varphi|_L=\varphi(L)$ and $\ker\psi\subseteq\varphi(L)$.

"strictness": The case of $\varphi$ is analogous, we prove it for $\psi$:

$f\in\mathcal{F}_\gamma N\cap\psi(M)$, with $f\notin\mathcal{F}_{\gamma^*}N$ $\Rightarrow$ $\exists\, m\in\mathcal{F}_\omega M$ with $\psi(m)=f$.

Since $\psi$ is $\Gamma$-filtered $\Rightarrow$ $\psi(\mathcal{F}_\omega M)\subseteq\mathcal{F}_\omega N$. $\Rightarrow$ $\omega\ge\gamma$.

If $\omega=\gamma$, we're done, since $f=\psi(m)\in\psi(\mathcal{F}_\gamma M)$.

Otherwise $\omega>\gamma$ and $(gr\,\psi)([m])=(gr\,\psi)(m+\mathcal{F}_{\omega^*}M)=\psi(m)+\mathcal{F}_{\omega^*}N$.

Since $\omega>\gamma$, we have $\psi(m)\in\mathcal{F}_\gamma N\subseteq\mathcal{F}_{\omega^*}N$ $\Rightarrow$ $[\psi(m)]=0$.

The exactness of $(gr\,*)$ implies the existence of $\ell\in\mathcal{F}_\alpha L$, s.t. $[m]=[\varphi(\ell)]$ and the $\Gamma$-filteredness of $\varphi$ implies $\alpha\ge\omega$. If $\alpha>\omega$ then the highest part of $\ell$ has to be in $\ker(\varphi)$.

$\Rightarrow$ we can substract all kernel parts of $\ell$ in finitely many steps arrive to $l' \in \mathcal{F}_w L \setminus \mathcal{F}_{w*} L$. For $m' = m - \varphi(l)$ we have $\psi(m') = \psi(m) = f$. By construction: $m' \in \mathcal{F}_{w'} M$ with $m' = 0$ or $w' < w$. We can proceed by $w > w' > \ldots > \gamma$.

$\tilde{m} - m = \varphi(l) + \varphi(l') + \ldots \in \varphi(L) \subseteq \ker(\psi)$.

Since $\psi(m) \neq 0 \Rightarrow \tilde{m} \notin \varphi(L)$. At each step with $m^{(i)} \neq 0$ we terminate at exactly $\gamma$. Hence $f = \psi(m) = \psi(\tilde{m})$. □