Algorithmic Homological Algebra

Lecture notes

WS 2013/14

Mohamed Barakat

Fakultät für Mathematik, Universität Kaiserslautern,
67653 Kaiserslautern
barakat@mathematik.uni-kl.de

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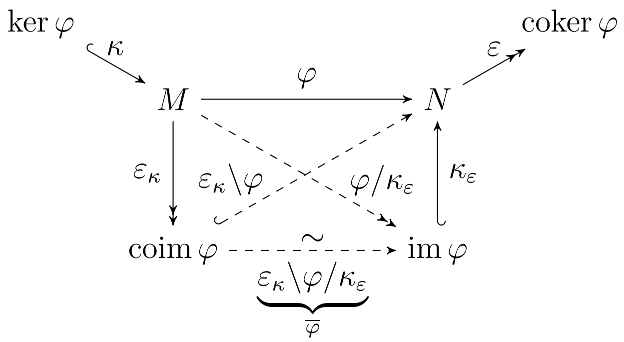

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Vorwort

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URL: http://www.mathematik.uni-kl.de/~barakat/Lehre/WS13/HomologicalAlgebra/Skript/AHA.pdf

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barakat@mathematik.uni-kl.de

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CHAPTER 1

Modules

1.1. Rings

If not further specified R will denote a ring with one which is *not* necessarily commutative. We assume ring homomorphisms to be unitial. A ring in which $1 = 0$ is called the[1] **zero ring**.

Example 1.1.1. Most of the following rings turn out to be "computable" in a sense which we will make precise later.

- *His majesty*, the ring $\mathbb{Z}$ of (rational) integers.
- There are several operations to obtain new rings from old ones:
  - **Residue class ring** R/I by a two-sided ideal $I \trianglelefteq R$ (e.g., the finite prime fields $\mathbb{F}_p = \mathbb{Z}/\langle p\rangle$ for a rational prime p, or $\mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{Z}$).
  - **Localizations** of commutative rings:
    - **Field of fractions** $\mathrm{Frac}(R)$ of a domain R (e.g., the field of rational numbers $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$).
    - **Localization** $S^{-1}R$ at a finitely generated multiplicative set $S \subset R$. For $f \in R$ and $S = \{1, f, f^2, \dots\}$ we write $R_f = R[\frac{1}{f}] := S^{-1}R$ (e.g., $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$).
    - **Localization** $R_\mathfrak{m}$ at maximal[2] ideals $\mathfrak{m} \lhd R$ (e.g., $\mathbb{Z}_{\langle p\rangle} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\}$).
  - **ORE extensions**:
    - **Polynomial ring** $R[x_1, \dots, x_n]$.
    - **Differential ring** $R[x, \frac{\mathrm{d}}{\mathrm{d}x}]$.
    - ...
  - **Direct product** of rings $R \times S$:
    $$(a, b) \pm (a', b') = (a \pm a', b \pm b'), (a, b) \cdot (a', b') = (a \cdot a', b \cdot b').$$
  - **Tensor product** of commutative rings $R \otimes S$:
    $$\otimes \text{ is bilinear and } (a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb').$$
  - **Convolution rings** of finite categories:
    - **Group ring** $R[G] := \{f : G \to R\}$ of a finite group[3] G with pointwise addition (and subtraction) and multiplication defined by the convolution
      $$(f * f')(g) = \sum_{hk=g} f(h)f'(k).$$
    - ...
  - **Opposite ring** $(R^{\mathrm{op}}, +, *)$ of any ring $(R, +, \cdot)$ with the same underlying sets and the same addition but reversed multiplication $a * b := b \cdot a$.

So a ring like $R = \mathbb{Q}[i](a)[b, c, \frac{1}{c}][x, \frac{\mathrm{d}}{\mathrm{d}x}]$ will be computable in our context.

Exercise 1.1. Different sequences of different operations do not necessary lead to non-isomorphic rings. Let k be a field. By ζ, i we denote the third and fourth root of unity, respectively, and by C_n a cyclic group of order n. Prove that:

(a) $k[x, x^{-1}] := k[x][\frac{1}{x}] \cong k[x, y]/\langle xy - 1\rangle$. The ring $k[x, x^{-1}]$ is called the univariate **LAURENT polynomial ring**. Can you generalize this isomorphism?

---

[1]The zero ring is unique up to a unique isomorphism.

[2]Maximal ideals are very special prime ideals.

[3]Here we view G as a category with invertible arrows over a single object. If G is not finite we only consider functions with finite support.

(b) $k[x,y]/\langle x,y\rangle \cong k$ and $k[x]/\langle x^2-x\rangle \cong k\times k$.
(c) $k[C_2] \cong k[x]/\langle x^2-1\rangle \cong k\times k$ only if $\operatorname{char} k \neq 2$. What about $\mathbb{Q}[C_3]$ and $\mathbb{Q}(\zeta)[C_3]$?
(d) $\mathbb{Z}[x]/\langle x^2+1\rangle \cong \mathbb{Z}[i]$ and $\mathbb{Z}[i]\otimes_{\mathbb{Z}}\mathbb{Z}[i] \cong \mathbb{Z}[i][x]/\langle x^2+1\rangle \cong \mathbb{Z}[i]\times\mathbb{Z}[i]$.
(e) $\mathbb{Q}[C_4] \cong \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}[i]$, what about $\mathbb{Q}[i][C_4]$?
(f) $k[x]\otimes_k k[y] \cong k[x,y]$.
(g) $k[\mathbb{N}^k] \cong k[x_1,\ldots,x_k]$ where $\mathbb{N}^k$ is a semi-group with 0 as neutral element.
(h) Of course, if R is commutative then $R\cong R^{\mathrm{op}}$. Prove that for a group ring $R=k[G]$ the previous isomorphism still holds even if G is not ABELian.
(i) $\mathbb{Q}[x,\frac{\mathrm{d}}{\mathrm{d}x}]^{\mathrm{op}} \cong \mathbb{Q}[x,\frac{\mathrm{d}}{\mathrm{d}x}]$.
(j) Find an example of a noncommutative ring where $R\not\cong R^{\mathrm{op}}$.

Hints: The homomorphism theorem (for rings) and the Chinese Remainder Theorem.

Rings appear in nature as endomorphism (sub)rings of ABELian groups.

Main example 1.1.2. Let M be an ABELian group. The set of group endomorphisms $\operatorname{End}(M)$ is a ring with pointwise addition (and subtraction) and multiplication given by the composition: $(f\circ f')(g) = f(f'(g))$. Here we apply functions from the left. The opposite ring $\operatorname{End}(R)^{\mathrm{op}}$ corresponds to the convention of applying functions from the right. Please get used to this as we will use both conventions!

Relating an abstract ring R with an endomorphism ring $\operatorname{End}(M)$ of an ABELian group M is the starting point of module theory.

1.2. The category of modules

Rings are, like groups and varieties, highly nonlinear objects. Such non-linear objects can be studied in terms of the category of all their linear "approximations". These associated linear objects are usually called modules.

1.2.a. Objects.

Because of the possible noncommutativity of rings we have to distinguish[4] between left and right modules.

Definition 1.2.1. A **left R-module** M is an ABELian group on which R acts by endomorphisms. This action is given by a ring homomorphism $\rho : R\to \operatorname{End}(M)$, called a **left representation** of R. A **right representation** $\rho : R\to\operatorname{End}(M)^{\mathrm{op}}$ defines a **right R-module**. We occasionally denote the left (resp. right) R-module M by ${}_RM$ (resp. M_R).

As usual, the left representation ρ can be interpreted as the CURRYing[5] of a left **action map**

$$\alpha_\rho : R\times M\to M, (r,m)\mapsto rm,$$

which is bilinear, associative, and satisfies $1m=m$ for all $m\in M$. In other words, $\alpha_\rho(r,m) = \rho(r)(m)$. Likewise for right representations and right action maps $\alpha_\rho : M\times R\to M, (m,r)\mapsto mr$.

Example 1.2.2. The following examples are immediate:

(a) A $\mathbb{Z}$-module is the same as an ABELian group.
(b) A k-module over a (skew)field k is the same as a k vector space.
(c) A trivial group $\{0\}$ is an R-module for any ring R. We call it a **zero module**.
(d) The multiplication map of the ring R endows R with a left and with a right module structure, which we denote by ${}_RR$ and R_R, respectively.
(e) Any ABELian group is a left $\operatorname{End}(M)$-module and a right $\operatorname{End}(M)^{\mathrm{op}}$-module.
(f) A right R-module is the same as a left R^{op}-module.

[4]This is also important for naturality questions.
[5]After the American mathematician and logician HASKELL BROOKS CURRY.

(g) If we identify ring elements with 1×1-matrices then the matrix multiplications $R^{1\times 1} \times R^{1\times c} \to R^{1\times c}$ and $R^{r\times 1} \times R^{1\times 1} \to R^{r\times 1}$ define a left R-module structure on the row space $R^{1\times c}$ and a right R-module structure on the column space[6] $R^{r\times 1}$.

1.2.b. Morphisms. Immediately after defining a mathematical structure and providing some examples one should define the structure preserving maps:

Definition 1.2.3. A group homomorphism $\varphi : M \to N, m \mapsto m\varphi$ of two left R-modules M, N is called an **R-module homomorphism, or an R-module map**, or **R-map** for short, if it is compatible[7] with the action of R, i.e., $(rm)\varphi = r(m\varphi)$ for all $r \in R, m \in M$. For right modules we write $\varphi : M \to N, m \mapsto \varphi m$ and require that $\varphi(mr) = (\varphi m)r$.

End lecture 1

The set $\mathrm{Hom}_R(M, N)$ of all R-maps from M to N with pointwise addition is an ABELian group, called the **homomorphism group** of M and N. We occasionally drop the index R when it is clear from the context. The **zero map** $0_{MN} : M \to N$ is the neutral element of this group.

The ABELian group $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ of all R-self-maps with composition is a ring, called the **endomorphism ring** of M. The **identity map** 1_M is the unit of this ring.

REMARK 1.2.4. Here we use the so-called **associative convention** for both right *and* left modules. It differs from the usual "mixed" convention, where the so-called commutative convention is used for left modules and the associative for right ones. The mixed convention is unavoidable if one insists on applying R-maps from the left, regardless of the parity of the modules. There one would write for left R-maps $\varphi : M \to N, m \mapsto \varphi(m)$ with $\varphi(rm) = r\varphi(m)$ and say that φ *commutes* with the action of R.

The associativity convention has at least two advantages:

(a) It is a symmetric convention: $\mathrm{End}(R_R) \cong R \cong \mathrm{End}({}_RR)$.
(b) It is compatible with matrix multiplication and therefore computer friendly:

$$\mathrm{Hom}({}_RR^{1\times c}, {}_RR^{1\times c'}) \cong R^{c\times c'} \text{ and } \mathrm{Hom}(R_R^{r\times 1}, R_R^{r'\times 1}) \cong R^{r'\times r}$$

and the associativity rules $(rm)\varphi = r(m\varphi)$ and $\varphi(mr) = (\varphi m)r$ follow from the associativity of matrix multiplication.

The mixed convention breaks the above symmetry: $\mathrm{End}(R_R) \cong R$ but $\mathrm{End}({}_RR) \cong R^{\mathrm{op}}$. It also makes working out the relation between applying R-maps in $\mathrm{Hom}({}_RR^{1\times c}, {}_RR^{1\times c'})$ and matrix multiplication a nontrivial exercise.

Convention. From now on we won't specify the parity unless necessary. If the formulation of some statement does depend on the parity then we will formulate it for *left* modules. The case of right modules is then analogous. As a consequence we will silently apply R-maps from the right. In particular, in the composition $\varphi\psi$ we first apply the *left* map φ, then ψ.

By replacing definitions which refer to elements by ones which exclusively refer to maps we will gradually replace the set-theoretic language by the categorical one. Doing so we can already make first steps in category theory, even before encountering the abstract definition of a category. The categorical approach, although more cumbersome at the beginning, will help us to reduce many known constructions to few basic ones. This will be very rewarding when we start dealing with computability as it will streamline all our algorithms. To emphasize this "element-less" approach we will replace the words "module" by "object" and "map" by "morphism".

An R-map is called **injective**, **surjective**, or **bijective**, if the underlying set-theoretic map is so. We will make an attempt to describe these properties using maps (or, better, morphisms) exclusively.

Definition 1.2.5 ([HS97, §II.3, p. 48])**.**

[6]Like SINGULAR, the subsystem PLURAL only uses column spaces as a data structure for modules but treats them as left modules (maybe because algorithms in the noncommutative GRÖBNER basis literature are usually developed for left ideals). This results in a mess when the ring is noncommutative.

[7]We deliberately drop the brackets $(m)\varphi$ for the associativity appearance of the law.

(epi) A morphism $\pi : M \to N$ is an **epimorphism** (or **epi**, or **epic**) if it is **pre-cancelable**, i.e.,

$$\pi\varphi = \pi\psi \Longrightarrow \varphi = \psi$$

for all objects L and all morphisms $\varphi, \psi \in \operatorname{Hom}(N, L)$. We write $\pi : M \twoheadrightarrow N$.

(mono) A morphism $\iota : M \to N$ is a **monomorphism** (or **mono**, or **monic**) if it is **post-cancelable**, i.e.,

$$\varphi\iota = \psi\iota \Longrightarrow \varphi = \psi$$

for all objects L and all morphisms $\varphi, \psi \in \operatorname{Hom}(L, M)$. We write $\iota : M \hookrightarrow N$.

Exercise 1.2. Prove the following statements for an R-map $\varphi : M \to N$.

- φ is injective iff it is mono.
- φ is surjective iff it is epi.
- φ is an bijective iff it is both mono and epi.

Although the previous definition is the correct categorical one it may look awkward at the beginning. Motivated by the underlying set-theoretic maps and using the axiom of choice one might find the following definition more "natural".

Definition 1.2.6.

(a) A morphism $\pi : M \to N$ is a **split epi** if it $\boxed{\text{has}}$ a **pre-inverse** (or **section**), i.e., a morphism $M \overset{\sigma}{\leftarrow} N$ such that

$$\sigma\pi = 1_N.$$

(b) A morphism $\iota : M \to N$ is a **split mono** if it $\boxed{\text{has}}$ a **post-inverse** (or **complement**), i.e., a morphism $M \overset{\chi}{\leftarrow} N$ such that

$$\iota\chi = 1_M.$$

(c) A morphism $\alpha : M \to N$ is an **isomorphism** (or **iso**) if it is a split epi and a split mono. We write $\alpha : M \overset{\cong}{\to} N$, or $\alpha : M \overset{\sim}{\to} N$, or $\alpha : M \cong N$, or $M \cong_\alpha N$. We call two objects M, N **isomorphic** if there exists an isomorphism $\alpha : M \overset{\sim}{\to} N$.

Here is an easy exercise:

Exercise 1.3. Prove without using elements:

(a) A split epi is an epi.
(b) A split mono is a mono.
(c) The pre- and post-inverse of an isomorphism coincide, and are hence unique. We call it **the inverse**.
(d) The composition of two (split) monos is a (split) mono.
(e) The composition of two (split) epis is a (split) epi.
(f) The composition of isomorphisms is an isomorphism.
(g) If $\varphi\psi$ is a mono then φ is a mono. The converse is false.
(h) If $\varphi\psi$ is an epi then ψ is an epi. The converse is false.

Show that in the sequence of $\mathbb{Z}$-maps $\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ the left morphism is mono, the right is epi, and non of them is split (as $\mathbb{Z}$-maps). If we view them as maps between sets then both become split.

As we saw above, an R-map which is epi and mono is bijective. It is easy to see that the inverse is again an R-map so we have an iso. Later we will encounter categories (with modules as objects!) in which being mono and epi does not imply being iso.

1.2.c. Subobjects and factor objects.

Definition 1.2.7 (Set-theoretic definition)**.** Let M be an R-module. An R**-submodule** N is a subgroup of M which is stable under the action of R. We write $N \leq_R M$. If R is clear from the context we just say submodule and write $N \leq M$. The (set theoretic) **embedding** $\iota : N \hookrightarrow M$ is a monic R-map. A submodule N is called **proper** if $\{0\} \neq N \neq M$. A module M with no proper submodules is called **simple**.

Example 1.2.8. Submodules of ${}_RR$ are the **left ideals** and of R_R the **right ideals**. A (skew)field k is a simple k-module.

Example 1.2.9. Let $\varphi : M \to N$ be an R-map.

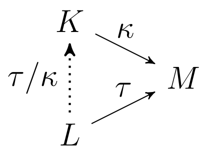
- The set-theoretic **image** $\operatorname{im}\varphi$ is a submodule of N. The corestriction $\varphi : M \to \operatorname{im}\varphi$ is an R-epi.
- The group theoretic **kernel** $\ker\varphi$ is a submodule of M. The embedding $\kappa : \ker\varphi \to M$ is an R-mono.

An R-map $\varphi : M \to N$ is surjective if, by definition, $\operatorname{im}\varphi = N$. From group theory (applied to the underlying group theoretic map) we know that φ is injective iff $\ker\varphi = \{0\}$.

Definition 1.2.10. Let N be an R-submodule of M. The action of R on M induces an action on factor group M/N turning it into an R-module. We call it the **factor module** of M modulo N. The **projection** $\varphi : M \to M/N, m \mapsto m + N$ is an R-epi.

Example 1.2.11. Let $\varphi : M \to N$ be an R-map.

- The **cokernel** $\operatorname{coker}\varphi := N/\operatorname{im}\varphi$ is a factor module of N.
- The **co-image** $\operatorname{coim}\varphi := M/\ker\varphi$ is a factor module of M.



Definition 1.2.12. We say that a morphism $\kappa : K \to M$ **dominates** $\tau : L \to M$ if κ is a **post-factor** of τ, i.e., if there exists a morphism $\tau/\kappa : L \to K$ such that $(\tau/\kappa)\kappa = \tau$. Any such morphism τ/κ is called a **lift** of τ along κ.

It is an easy exercise to prove that mutual dominance is an equivalence relation and that two R-maps which are mutually dominant have the same image submodule. The converse of the last statement is true for monos. In other words, we can identify a submodule $K \leq M$ with the class of all R-monos having K as their image.

This is how we can redefine submodules categorically:

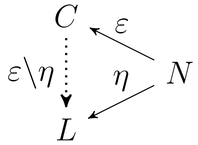
Definition 1.2.13. A **subobject** $K \leq M$ is an equivalence class of *monos* with M as target under the equivalence relation of mutual dominance.

One can also define the **poset** of submodules categorically:

Definition 1.2.14. For two subobjects $L, K \leq M$ represented by $L \overset{\tau}{\hookrightarrow} M \overset{\kappa}{\hookleftarrow} K$ we say that L is **smaller** than K and write[8] $L \leq K$ (or K is **larger** than L and write $K \geq L$) if κ dominates τ.

We could now ask if we can redefine the entire **lattice** of submodules, i.e., the sum and intersection of two submodules, categorically. The answer is yes, but we first need a categorical definition of kernels, cokernels, and direct sums, which we will treat in the following subsections.

Before ending this subsection we dualize the above definition and get:



Definition 1.2.15. We say that a morphism $\varepsilon : N \to C$ **codominates** $\eta : N \to L$ if ε is a **pre-factor** of η, i.e., if there exists a morphism $\varepsilon\backslash\eta : C \to L$ such that $\varepsilon(\varepsilon\backslash\eta) = \eta$. Any such morphism $\varepsilon\backslash\eta$ is called a **colift** of η along ε.

Dually, this leads to a categorical formulation of a **factor object**.

REMARK 1.2.16. Lifts along monos and colifts along epis are unique.

PROOF. This is immediate from the definitions of mono and epi. □

Exercise 1.4. Let $L \leq K \leq M$ be two subobjects of M represented by $L \overset{\tau}{\hookrightarrow} M \overset{\kappa}{\hookleftarrow} K$. Show that the unique lift $L \overset{\tau/\kappa}{\hookrightarrow} K$ is a mono realizing L as a subobject of K, justifying the subobject notation $L \leq K$.
Hint: Exercise 1.3.(g).

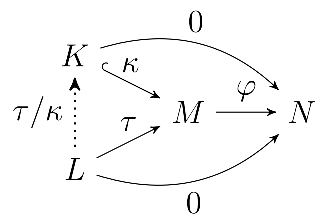
[8] Exercise 1.4 will justify notation which was reserved for subobjects.

1.2.d. Kernels and cokernels, images and co-images. We now formulate so-called **universal properties** of the previous set-theoretic constructions, which will server as categorical characterizations. I will leave it as a general exercise to prove that the categorical definitions coincide with the module theoretic ones.

Definition 1.2.17 ([HS97, §II.6, p. 61])**.** Let $\varphi : M \to N$ be a morphism.

(ker) "The" **kernel** of φ is an object K, usually denoted by $\ker \varphi$, *together* with morphism $\kappa : K \to M$ satisfying the following universal property
  (i) $\kappa\varphi = 0$, and
  (ii) κ dominates all such morphisms, i.e., for all objects L and all morphisms $\tau : L \to M$ with $\tau\varphi = 0$ there exists a *unique* lift $\tau/\kappa : L \to K$ (of τ along κ with $\tau = (\tau/\kappa)\kappa$).

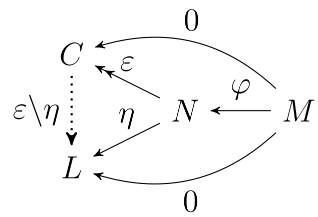
It follows from the uniqueness of the lift τ/κ that κ is a *mono*[9]. We will usually call it the **kernel mono**.



K is called "the" **kernel object** of φ. Depending on the context $\ker \varphi$ sometimes stands for the morphism κ and sometimes for the object K.

(coker) "The" **cokernel** of φ is an object C, usually denoted by $\operatorname{coker} \varphi$, *together* with a morphism $\varepsilon : M \to C$ satisfying the following universal property
  (i) $\varphi\varepsilon = 0$, and
  (ii) ε codominates all such morphism, i.e., for all objects L and all morphisms $\eta : N \to L$ with $\varphi\eta = 0$ there exists a *unique* colift $\varepsilon\backslash\eta : C \to L$ (of η along ε with $\eta = \varepsilon(\varepsilon\backslash\eta)$).

It follows from the uniqueness of the colift $\varepsilon\backslash\eta$ that ε is an *epi*. We will usually call it the **cokernel epi**.



C is called "the" **cokernel object** of φ. Depending on the context $\operatorname{coker} \varphi$ sometimes stands for the morphism ε and sometimes for the object C.

Remark 1.2.16 implies that instead of insisting on the uniqueness of the lift and the colift in the definition of kernels and cokernels we could have equivalentely required that κ is mono and ε is epi. Luckily, this is already enough to define images and coimages without referring to elements.

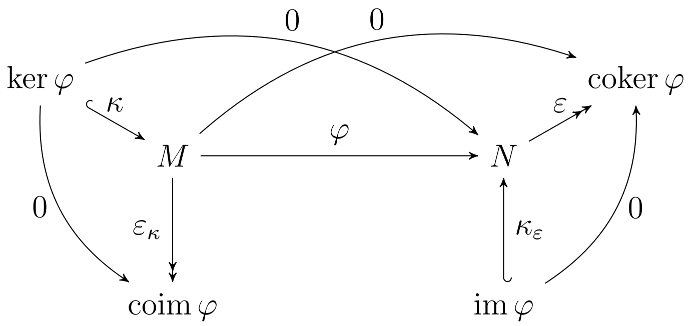
End lecture 2

Definition 1.2.18. Let $\varphi : M \to N$ be a morphism.

(im) The **image** of φ, denoted by $\operatorname{im} \varphi$, is the kernel of its cokernel epi.

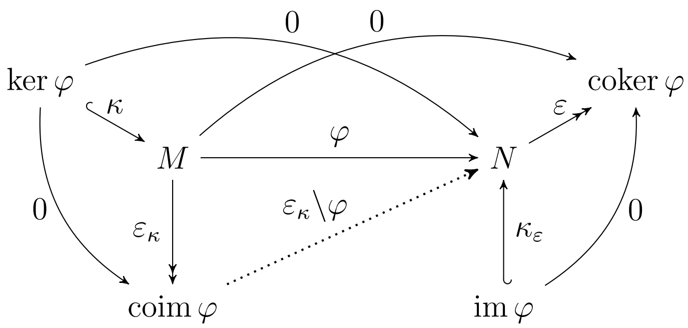
(coim) The **co-image** of φ, denoted by $\operatorname{coim} \varphi$, is the cokernel of its kernel mono.

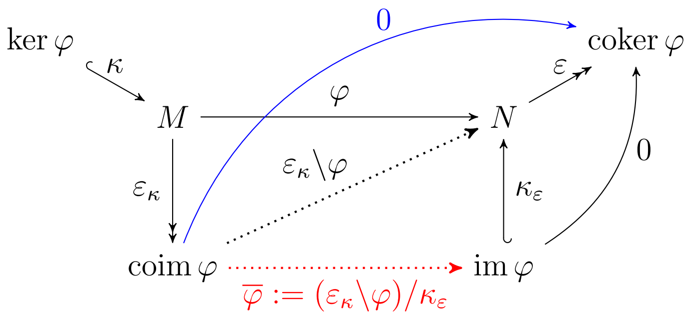
[9] Since $(\alpha\kappa)\varphi = \alpha(\kappa\varphi) = 0$ we can reconstruct α as the unique lift of $\alpha\kappa$ along κ.



We denote the cokernel epi of the kernel mono κ by ε_κ and call it the **co-image epi**. Dually, we denote the kernel mono of the cokernel epi ε by κ_ε and call it the **image mono**.

The homomorphism theorem for R-modules states that each R-map φ induces an R-map $\overline{\varphi}$: $\operatorname{coim}\varphi \to \operatorname{im}\varphi$ (between its coimage and its image) which is an isomorphism. Now we construct this induced morphism categorically by using the above diagrams:





Do we get the same morphism $\overline{\varphi}$ if we reverse the process, i.e., first lift then colift: $\varepsilon_\kappa \backslash (\varphi/\kappa_\varepsilon)$? The answer is yes since for both of them pre-composing the *epi* ε_κ and post-composing the *mono* κ_ε yields φ. Hence, both are equal.

Summing up, in categories in which cokernels and kernels exist we have **categorical algorithms** to compute coimages and images together with an induced morphism $\overline{\varphi}$: $\operatorname{coim}\varphi \to \operatorname{im}\varphi$. This morphism is an isomorphism is the category of modules but not an isomorphism in general categories with kernels and cokernels. However, being an isomorphism is one of the defining axioms of so-called ABELian categories. Hence, in ABELian categories **the homomorphism theorem**, also called **the first isomorphism theorem**, will hold true.

In order to talk about the second isomorphism theorem we still need to be able to perform intersections and sums of subobjects categorically. The last missing ingredient are direct sums (more precisely, products and coproduct).

1.2.e. Direct sums and direct products. We start by recalling the set theoretic definition.

Definition 1.2.19 ([HS97, §II.5, p. 58, p. 54])**.** Let $\{A_i\}_{i \in I}$ be a family for R-modules over the index set I.

(prod) The set

$$\prod_{i \in I} A_i := \left\{ a : I \to \bigcup A_i \mid a_i := a(i) \in A_i \text{ for all } i \in I \right\}$$

is an R-module, called the **(direct) product** of $\{A_i\}_{i\in I}$. For each $i \in I$ we obtain an R-epi $\pi_i : \prod A_j \to A_i, a \mapsto a_i$, called the i-th **projection**.

(sum) The set

$$\bigoplus_{i\in I} A_i := \left\{ a \in \prod_{i\in I} A_i \mid a_i = 0 \text{ for all but finitely many } i \in I \right\}$$

is an R-module, called the **direct sum** or **coproduct** of $\{A_i\}_{i\in I}$. For each $i \in I$, we obtain an R-mono $\iota_i : A_i \hookrightarrow \bigoplus A_j, a_i \mapsto \left(j \mapsto \begin{cases} a_i & j = i \\ 0 & j \neq i \end{cases} \right)$, called the i-th **embedding**.

For a finite index set I it is clear that products and coproducts of modules coincide (this is false in general categories but still true in ABELian ones). We refer to them as "direct sum" and write $M \oplus N$. Although we will only deal with finite index sets giving the general definition was not more complicated. Now we give the categorical characterization:

Definition 1.2.20 ([**HS97**, §II.5, p. 58, p. 54])**.** Let I be an index set and $\{A_i\}_{i\in I}$ a family of objects.

(prod) "The" **product** of $\{A_i\}_{i\in I}$ is an object $\prod_{i\in I} A_i$ together with a family of morphisms $\{\pi_i : \prod A_j \twoheadrightarrow A_i\}_{i\in I}$, called **projections**[10], such that the following universal property is satisfied:
For any object M and any family $\{\varphi_i : M \to A_i\}_{i\in I}$ of morphisms there $\boxed{\text{exists}}$ a *unique* morphism

$$\varphi = \{\varphi_i\} : M \to \prod A_i,$$

called the **product morphism** or **pairing**, such that $\varphi\pi_i = \varphi_i$ for all $i \in I$.

(sum) "The" **coproduct** of $\{A_i\}_{i\in I}$ is an object $\bigoplus_{i\in I} A_i$ together with a family of morphisms $\{\iota_i : A_i \hookrightarrow \bigoplus A_j\}_{i\in I}$, called **embeddings**[11], such that the following universal property is satisfied:
For any object M and any family $\{\varphi_i : A_i \to M\}_{i\in I}$ of morphisms there $\boxed{\text{exists}}$ a *unique* morphism

$$\varphi = \langle\varphi_i\rangle : \bigoplus A_i \to M,$$

called the **coproduct morphism** or **copairing**, such that $\iota_i\varphi = \varphi_i$ for all $i \in I$.

The universal properties imply that the product (resp. coproduct), in case it exists, is *unique up to a unique isomorphism*.

REMARK 1.2.21.

(prod) The product morphism for a family $\{\varphi_i : M \to A_i\}_{i\in I}$ of R-maps is given by

$$\varphi = \{\varphi_i\} : M \to \prod A_i, m \mapsto (i \mapsto m\varphi_i).$$

(sum) The coproduct morphism for a family $\{\varphi_i : A_i \to M\}_{i\in I}$ of R-maps is given by

$$\varphi = \langle\varphi_i\rangle : \bigoplus A_i \to M, a \mapsto \sum_{i\in I} a_i\varphi_i.$$

1.2.f. Pull-backs and push-outs as kernels and cokernels. In this subsection we start with the categorical definition and reduce it to the computation of kernels of coproduct morphisms and cokernels of product morphisms. This also gives a module theoretic definition.

Definition 1.2.22.

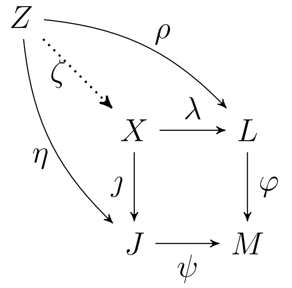
(pull) A **pull-back** of two morphisms $L \stackrel{\varphi}{\to} M \stackrel{\psi}{\leftarrow} J$ is an object X together with a pair of morphisms $L \stackrel{\lambda}{\leftarrow} X \stackrel{\jmath}{\to} J$, called the **pull-back morphisms**, completing the square (i.e., such that $\lambda\varphi = \jmath\psi$) and satisfying the following universal property:

[10] The projections are not necessarily epis unless the category has a zero object [**Bru**].

[11] The embeddings are not necessarily monos unless the category has a zero object [**Bru**].

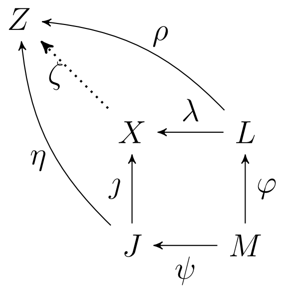
For any object Z and any pair of morphisms $L \overset{\rho}{\leftarrow} Z \overset{\eta}{\rightarrow} J$ completing the square there exists a *unique* $\zeta : Z \to X$ with $\rho = \zeta\lambda$ and $\eta = \zeta\jmath$.

(pull-back diagram)



(push) Reversing all arrows in the above definition we obtain the dual definition of the **push-out** X of two morphisms $L \overset{\varphi}{\leftarrow} M \overset{\psi}{\rightarrow} J$, together with two **push-out morphisms** $L \overset{\lambda}{\rightarrow} X \overset{\jmath}{\leftarrow} J$.

(push-out diagram)



Here we only considered *finite* pull-backs and push-outs. Next we show how a *finite* pull-back can be reconstructed as a kernel of a coproduct morphism and, dually, a *finite* push-out as a cokernel of a product morphism. We will make use of the fact that in our context finite products and coproducts coincide.

Proposition 1.2.23. *The following two dual statements hold.*

(pull) The pull-back of two morphisms $L \overset{\varphi}{\rightarrow} M \overset{\psi}{\leftarrow} J$ is the kernel $X \overset{\{\lambda,\jmath\}}{\hookrightarrow} L \oplus J$ of the coproduct *morphism $L \oplus J \xrightarrow{\langle\varphi,-\psi\rangle} M$.*

(push) The push-out of two morphisms $L \overset{\varphi}{\leftarrow} M \overset{\psi}{\rightarrow} J$ is the cokernel $X \overset{\langle\lambda,\jmath\rangle}{\twoheadleftarrow} L \oplus J$ of the product *morphism $L \oplus J \xleftarrow{\{\varphi,-\psi\}} M$.*

PROOF.

(pull) One can recover the pull-back morphisms $L \overset{\lambda}{\leftarrow} X \overset{\jmath}{\rightarrow} J$ by post-composing the kernel mono $\{\lambda, \jmath\}$ with the *projections* $L \overset{\pi_L}{\longleftarrow} L \oplus J \overset{\pi_J}{\longrightarrow} J$. This justifies the name $\{\lambda, \jmath\}$. The unique morphism ζ in the pull-back diagram is the lift of the product morphism $\{\rho, \eta\}$ along the kernel mono $\{\lambda, \jmath\}$.

(push) Dually, one can recover the push-out morphisms $L \overset{\lambda}{\rightarrow} X \overset{\jmath}{\leftarrow} J$ by pre-composing the cokernel epi $\langle\lambda, \jmath\rangle$ with the *embeddings* $L \overset{\iota_L}{\longrightarrow} L \oplus J \overset{\iota_J}{\longleftarrow} J$. This justifies the name $\langle\lambda, \jmath\rangle$. The unique morphism ζ in the push-out diagram is the colift of of the coproduct morphism $\langle\rho, \eta\rangle$ along the cokernel epi $\langle\lambda, \jmath\rangle$. □

Now we are finally able to define the intersection of two submodules categorically. I leave the sum as the dual exercise.

Definition 1.2.24. The intersection of two subobjects $L \overset{\iota_L}{\hookrightarrow} M \overset{\iota_J}{\hookleftarrow} J$ is their pull-back.

REMARK 1.2.25. The module (or set) theoretic definition of intersection coincides with the above categorical one.

PROOF. The proof uses the reconstruction of the pull-back as the kernel of the coproduct morphism. Indeed:

$$\ker\langle\iota_L, -\iota_J\rangle = \{(m, n) \in L \oplus J \mid 0 = \iota_L(m) - \iota_J(n) = m - n\} \cong L \cap J. \qquad \square$$

End lecture 3

Before we see how to describe a constructive setup for all this and more it is time to define ABELian categories.

CHAPTER 2

ABELian categories

2.1. Categories

The notion of a category is a special case of what is nowadays called a **horizontal categorification** or **oidification** of the notion of a **monoid**. Another funny name would be "monoidoid". This will become clear later.

Definition 2.1.1. A **quiver** (or **multi-digraph**) $\mathcal{A}$ consists of a class of **objects** $\mathcal{A}_0 = \operatorname{Obj}\mathcal{A}$ and a class of **morphisms** (or **arrows**) $\mathcal{A}_1 = \operatorname{Mor}\mathcal{A}$ together with two defining maps

$$s, t : \mathcal{A}_1 \rightrightarrows \mathcal{A}_0,$$

called **source** and **target**, respectively.

We write $\operatorname{Hom}_{\mathcal{A}}(M, N)$ (sometimes also $\mathcal{A}(M, N)$) for the fiber $(s,t)^{-1}(\{(M, N)\})$ of the product map $(s, t) : \mathcal{A}_1 \to \mathcal{A}_0 \times \mathcal{A}_0$ over the pair $(M, N) \in \mathcal{A}_0 \times \mathcal{A}_0$. This is the class of all morphisms φ with source $s(\varphi) = M$ and target $t(\varphi) = N$. We indicate this by writing $\varphi : M \to N$ or $M \xrightarrow{\varphi} N$. Hence, $\mathcal{A}_1$ is the disjoint union $\mathcal{A}_1 = \dot{\bigcup}_{(M,N)\in\mathcal{A}_0\times\mathcal{A}_0} \operatorname{Hom}_{\mathcal{A}}(M, N)$. As usual $\operatorname{End}_{\mathcal{A}}(M) := \operatorname{Hom}_{\mathcal{A}}(M, M)$. If the class $\operatorname{Hom}_{\mathcal{A}}(M, N)$ is a *set* for all pairs (M, N) then we call the quiver **locally small**. From now on all quivers will be assumed locally small. We therefore talk about the Hom**-sets**. If additionally $\mathcal{A}_0$ is a set then the quiver is called **small**.

Definition 2.1.2. A **category** $\mathcal{A}$ is a quiver with two further defining maps

$$\mathcal{A}_0 \xrightarrow{1} \mathcal{A}_1 \xleftarrow{\mu} \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1,$$

called **identity** and **composition**, respectively, subject to a list of defining properties.

To express these properties in a suggestive way we first introduce some notational conventions:

- The notation used in $\mu : \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_1$ means that μ is only defined for pairs of morphisms (φ, ψ) such that $t(\varphi) = s(\psi)$. We call such a pair **composable**. If not stated otherwise, we write $\varphi\psi$ instead of $\mu(\varphi, \psi)$. We call this the **left convention**, where the **right convention**[1] would be to write $\psi\varphi$ instead of $\varphi\psi$. In any case we call φ the **pre-morphism** and ψ the **post-morphism**.
- We use latin letters for objects and greek letters for morphisms, so $M \in \mathcal{A}$ and $\varphi \in \mathcal{A}$ means $M \in \mathcal{A}_0$ and $\varphi \in \mathcal{A}_1$. We also refer to them as $\mathcal{A}$**-objects** and $\mathcal{A}$**-morphisms**.
- We write 1_M for $1(M)$.

We now state the defining properties:

(a) $s(1_M) = t(1_M) = M$, i.e., $1_M \in \operatorname{End}_A(M)$ for all $M \in \mathcal{A}$.
In words: the identity respects source and target.

(b) $s(\varphi\psi) = s(\varphi)$ and $t(\varphi\psi) = t(\psi)$ for all composable morphism $\varphi, \psi \in \mathcal{A}$.
In words: the composition respects source and target.
We express this as follows:

$$\begin{array}{rcccl} \mu : \operatorname{Hom}_{\mathcal{A}}(M, L) & \times & \operatorname{Hom}_{\mathcal{A}}(L, N) & \to & \operatorname{Hom}_{\mathcal{A}}(M, N), \\ (\varphi & , & \psi) & \mapsto & \varphi\psi \end{array}$$

(c) $(\varphi\psi)\rho = \varphi(\psi\rho)$ for all composable morphisms $\varphi, \psi, \rho \in \mathcal{A}$.
In words: The composition is associative.

(d) $1_M\varphi = \varphi$ and $\psi 1_M = \psi$ for all φ with $s(\varphi) = M$ and all ψ with $t(\psi) = M$.
In words: The identity is a left and right unit of the composition.

[1] This is the usual convention followed in [HS97, §II.1, p. 41].

REMARK 2.1.3.

- The objects are in bijection to the identities via $M \mapsto 1_M$. So for many further constructions we only need to refer to morphisms.
- It follows from the axioms that the set $\operatorname{End}(M)$ is a monoid for all $M \in \mathcal{A}$. In particular, a category with one object is nothing but a monoid. This is what we mean by saying that "a category is a horizontally categorified monoid", or a "monoidoid".
- Isomorphic objects in a category need not be equal. In fact, this is one of the reasons for the flexibility of the notion of a category. A category in which isomorphic objects in $\mathcal{A}$ are equal is called **skeletal**.

Example 2.1.4.

(a) The category (Sets) with sets as objects and maps as morphisms.
(b) The category (Sets_0) with pointed sets as objects and maps preserving the distinguished points as morphisms.
(c) The category (Grps) with groups as objects and group homomorphisms as morphisms.
(d) The category $(\text{Ab}) = \mathbb{Z}-\mathrm{Mod}$ with ABELian groups as objects and group homomorphisms as morphisms.
(e) The category (Rngs) with (not necessarily unitial) rings as objects and ring homomorphisms (not necessarily respecting one) as morphisms.
(f) The category (uRngs) with unitial rings as objects and ring homomorphisms (not necessarily respecting one) as morphisms.
(g) The category (URngs) with unitial rings as objects and unitial ring homomorphisms as morphisms.
(h) The category (Top) with topological spaces as objects and continuous maps as morphisms.
(i) The category (Top_0) with pointed topological spaces as objects and continuous maps preserving the distinguished points as morphisms.
(j) The category of truth values with objects $\{\top, \bot\}$ and a single non-identity morphism $\bot \to \top$ saying that `false` implies `true`.
(k) A **proset** (=preordered set) is a **thin** small category, i.e., with at most one morphism between two objects.
(l) A **poset** (=partially ordered set) is a skeletal thin small category.
(m) The **simplicial category** Δ with totally ordered sets $\underline{n} = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N} = \{0, 1, ...\}$ as objects and order-preserving maps ($i \leq j \implies f(i) \leq f(j)$) as morphisms.

Non of the above category names in round brackets is standard.

Like with monoids, to each category $\mathcal{A}$ one can define the so-called **opposite** category $\mathcal{A}^{\text{op}}$ with $\mathcal{A}_0^{\text{op}} := \mathcal{A}_0$, $\mathcal{A}_1^{\text{op}} := \mathcal{A}_1$, $1_{\mathcal{A}^{\text{op}}} = 1_{\mathcal{A}}$, $s_{\mathcal{A}^{\text{op}}} = t_{\mathcal{A}}$ and $t_{\mathcal{A}^{\text{op}}} = s_{\mathcal{A}}$ (i.e., $\operatorname{Hom}_{\mathcal{A}^{\text{op}}}(M, N) := \operatorname{Hom}_{\mathcal{A}}(N, M)$) and composition $\mu_{\mathcal{A}^{\text{op}}}(\varphi, \psi) := \mu_{\mathcal{A}}(\psi, \varphi)$. Clearly, double dualizing recovers $\mathcal{A} = (\mathcal{A}^{\text{op}})^{\text{op}}$.

Introducing the opposite category saves us half of our definitions:

- A (split) epi in $\mathcal{A}$ is (split) mono in $\mathcal{A}^{\text{op}}$.
- A colift in $\mathcal{A}$ is a lift in $\mathcal{A}^{\text{op}}$.
- A cokernel epi in $\mathcal{A}$ is a kernel mono in $\mathcal{A}^{\text{op}}$.
- A coproduct in $\mathcal{A}$ is a product in $\mathcal{A}^{\text{op}}$.
- A push-out in $\mathcal{A}$ is a pull-back in $\mathcal{A}^{\text{op}}$.

After introducing functors, natural transformations, and equivalences of categories we will be able to formulate statements like: The opposite category of (URngs) is equivalent to the category of affine schemes $(\text{AffSch}) \simeq (\text{URngs})^{\text{op}}$.

Definition 2.1.5. A **subcategory** $\mathcal{A}'$ of a category $\mathcal{A}$ consists of subclasses $\mathcal{A}'_0 \subset \mathcal{A}_0$ and $\mathcal{A}'_1 \subset \mathcal{A}_1$ to which all defining maps $1, s, t, \mu$ restrict. It is called **full** if $\operatorname{Hom}_{\mathcal{A}'}(M, N) = \operatorname{Hom}_{\mathcal{A}}(M, N)$ for all $M, N \in \mathcal{A}'$.

A subcategory is again a category.

Example 2.1.6.

(a) (Ab) is a full subcategory of (Grps).
(b) (uRngs) is a full subcategory of (Rngs).
(c) (URngs) is a non-full subcategory of (uRngs).
(d) ...

Definitions 1.2.5 and 1.2.6 of (split) epi, (split) mono, and isomorphism were already categorical.

Definition 2.1.7. A category in which every morphism is an isomorphism is called a **groupoid**.

Example 2.1.8.

(a) If $\mathcal{A}$ is a groupoid then $\operatorname{End}_{\mathcal{A}}(M)$ is a group for all objects $M \in \mathcal{A}$. Conversely, a group is a groupoid with one object. This is our second example of an oidification.
(b) The (generally non-full) subcategory $\mathcal{A}^*$ consisting of all isomorphisms in the category $\mathcal{A}$ is a groupoid, called the **groupoid of units**.
(c) An equivalence relations is a groupoid with at most one arrow from each object to another.
(d) A group right-action $t : \Omega \times G \to \Omega, (\omega, g) \mapsto \omega g$ defines a groupoid with target t and source $s = \pi_1 : \Omega \times G \to \Omega, (\omega, g) \mapsto \omega$.

Definition 2.1.9 ([**HS97**, §II.1, p. 43])**.** Let $\mathcal{A}$ be a category.

- A **terminal object** $T \in \mathcal{A}$ is an object such that $\operatorname{Hom}_{\mathcal{A}}(M, T)$ is a singleton, i.e., consists of a *single* morphism, for all $M \in \mathcal{A}$.
- An **initial object** $I \in \mathcal{A}$ is an object such that $\operatorname{Hom}_{\mathcal{A}}(I, M)$ is a singleton, i.e., consists of a *single* morphism, for all $M \in \mathcal{A}$.

Both terminologies are dual: T is terminal $\mathcal{A}$ iff T is initial in $\mathcal{A}^{\mathrm{op}}$. Trivially, initial objects and terminal objects are *unique up to unique isomorphisms* in $\mathcal{A}$. A terminal object is the same as an **empty product** and an initial the same as an **empty coproduct**.

Definition 2.1.10. A **zero object**[2] $0 \in \mathcal{A}$ is an object which is *both* initial and terminal[3]. A **category with zero** is a category that has a zero object[4].

Define for each pair of objects M, N the *unique* **zero morphism**

$$0_{MN} := M \to 0 \to N.$$

We write 0 instead of 0_{MN} when no confusion is possible.

Example 2.1.11.

(a) $\mathbb{Z}$ is an initial object in (URngs) which is not terminal.
(b) The zero ring is a terminal object in (URngs) which is not initial.
(c) The trivial group is a zero object in (Grps) and (Ab). Both are categories with zero.

2.2. Additive and ABELian categories

Definition 2.2.1. A category $\mathcal{A}$ is called **pre-additive** if it is **enriched over** (Ab), i.e., if for all objects $M, N, L \in \mathcal{A}$

(pAdd1) the set $\operatorname{Hom}_{\mathcal{A}}(M, N)$ is an (additively written) ABELian group, in particular, there exists an addition (and a subtraction) operation turning $\operatorname{Hom}_{\mathcal{A}}(M, N)$ into an ABELian group;
(pAdd2) the composition $\operatorname{Hom}_{\mathcal{A}}(M, N) \times \operatorname{Hom}_{\mathcal{A}}(N, L) \to \operatorname{Hom}_{\mathcal{A}}(M, L)$ is bilinear.

REMARK 2.2.2. Let $\mathcal{A}$ be a pre-additive category.

(a) The definition of pre-additive is self-dual, i.e., the opposite category $\mathcal{A}^{\mathrm{op}}$ is again pre-additive.

[2] also called **empty biproduct**.
[3] This is our first **self-dual** notion!
[4] This is a category enriched over the category of pointed sets.

(b) If $\mathcal{A}$ has a zero object then the unique zero morphism 0_{MN} defined above is exactly the zero element of the ABELian group $\operatorname{Hom}_{\mathcal{A}}(M, N)$, so there is no ambiguity talking about the zero morphism from M to N.

(c) The ABELian group $\operatorname{End}_{\mathcal{A}}(M)$ is a unitial ring for all $M \in \mathcal{A}$. Conversely, a unitial ring is a pre-additive category with one object. Another name for a pre-additive category would therefore be a "ringoid". This is our third and last example of an oidification.

Definition 1.2.19 of product and coproduct was already categorical.

Definition 2.2.3 ([**HS97**, §II.9, p. 75])**.** A *pre-additive* category *with zero* is called **additive** if

(Add) The *product*[5] $(M \oplus N; \pi_M, \pi_N)$ of two arbitrary objects M and N $\boxed{\text{exists}}$.

REMARK 2.2.4. The following is true for additive categories (cf. [**HS97**, §II.9, p. 75ff]):

- For objects M and N define the morphisms
$$\iota_M := \{1_M, 0_{MN}\} : M \to M \oplus N \quad \text{and} \quad \iota_N := \{0_{NM}, 1_N\} : N \to M \oplus N.$$
Then $\pi_M \iota_M + \pi_N \iota_N = 1_{M \oplus N}$.
- It follows that finite *coproducts* also exist: $(M \oplus N; \iota_M, \iota_N)$ with ι_M, ι_N as above and the coproduct morphism defined by
$$\langle \varphi, \psi \rangle := \pi_M \varphi + \pi_N \psi : M \oplus N \to L,$$
for two morphisms $\varphi : M \to L$ and $\psi : N \to L$. In additive categories one often uses the terminology **sum** instead of coproduct (or product).
- In particular, *finite* products and coproducts "coincide". Such objects are called **biproducts**.

We have just proved that the definition of an additive category is self-dual, i.e., the opposite $\mathcal{A}^{\mathrm{op}}$ of an additive category $\mathcal{A}$ is again additive.

- For $K \xrightarrow{\{\alpha,\beta\}} L \oplus M \xrightarrow{\langle \varphi, \psi \rangle} N$ we have $\{\alpha, \beta\}\langle \varphi, \psi \rangle = \alpha\varphi + \beta\psi$.
- In particular, for $\varphi, \psi : M \to N$ we have $\varphi + \psi = \{1_M, 1_M\}\langle \varphi, \psi \rangle$.

This means that if a category is additive, then its (in general non-unique) pre-additive structure is already determined by its structure as a category. In other words, there is then exactly one possible definition of the ABELian group structure on the Hom-sets. We say the addition for an additive category is internal to that category.

Definition 1.2.17 of kernel and cokernel is valid for any category with zero. Again, the following definition is self-dual.

Definition 2.2.5. An *additive* category is called **pre-ABELian** if every morphism

(pAb1) $\boxed{\text{has}}$ a kernel;

(pAb2) $\boxed{\text{has}}$ a cokernel.

Exercise 2.1. Let $\mathcal{A}$ be a pre-ABELian category with zero, kernels, and cokernels. Prove that for an $\mathcal{A}$-morphism φ:

- φ is mono iff $\ker \varphi = 0$;
- φ is epi iff $\operatorname{coker} \varphi = 0$.

Our definition of coimage and image and the construction of the induced morphism $\operatorname{coim} \varphi \xrightarrow{\overline{\varphi}} \operatorname{im} \varphi$ in §1.2.d are valid in any category with kernels and cokernels. Using this we can finally state the definition of an ABELian category:

Definition 2.2.6 ([**Rot09**, §5.5, p. 307])**.** A pre-ABELian category is called **ABELian** if the **homomorphism theorem** is valid, i.e., if for each morphism $\varphi \in \mathcal{A}$

(Ab) the induced morphism $\operatorname{coim} \varphi \xrightarrow{\overline{\varphi}} \operatorname{im} \varphi$ is an isomorphism.

End lecture 4

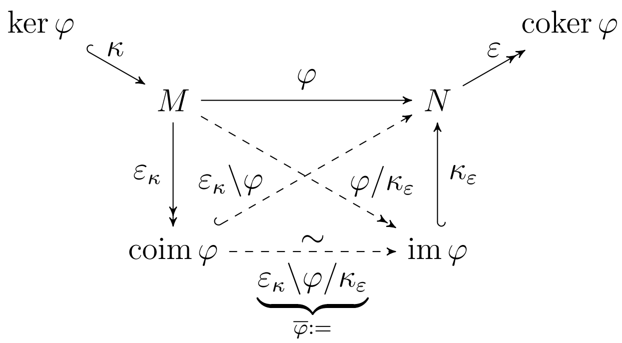
Note that this last axiom involves a *single* existential quantifier, namely the existence of an inverse. This is the first reason why we prefer it over other equivalent formulations. The second

[5]The use of the direct sum symbol $\oplus$ will be justified below.

reason is original axiom GROTHENDIECK's gave in his seminal TÔHOKU paper [Gro57]. The third reason is that it emphasizes the homomorphism theorem, which is in my opinion among the reasons behind various "tunnel effects" connecting apparently different subfields of mathematics.

Again, these definitions are self-dual, i.e., the opposite $\mathcal{A}^{\mathrm{op}}$ of a (pre-)ABELian category $\mathcal{A}$ is again (pre-)ABELian.

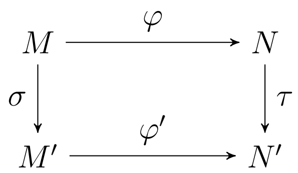
(Diag$_\varphi$)



From the identities $\varepsilon_\kappa\backslash\varphi = \overline{\varphi}\kappa_\varepsilon$ and $\varphi/\kappa_\varepsilon = \varepsilon_\kappa\overline{\varphi}$ we conclude that the colift $\varepsilon_\kappa\backslash\varphi$ is a mono (as the composition of an iso and a mono) and that lift $\varphi/\kappa_\varepsilon$ is an epi (as the composition of an epi and an iso).

Definition 2.2.7. We call the colift $\varepsilon_\kappa\backslash\varphi$ the **co-image mono**[6] and the lift $\varphi/\kappa_\varepsilon$ the **image epi**.

Exercise 2.2. Show that the above diagram is functorial, i.e., a commutative diagram of the form

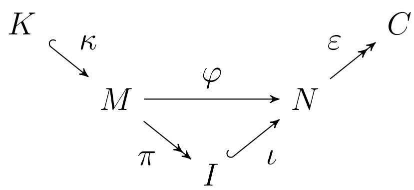


can be extended to a commutative diagram between (Diag$_\varphi$) and (Diag$_{\varphi'}$).

We summarize this in the following proposition.

Corollary 2.2.8 ([HS97, Prop. II.9.6],[ML95, Thm. IX.2.1])**.** *For each morphisms $\varphi : M \to N$ in an* ABEL*ian category there is a so-called kernel-cokernel sequence* (S$_\varphi$)

(S$_\varphi$)



where

- $\pi\iota = \varphi$*, i.e.,* φ *is the composition of an epi and a mono,*
- κ *is the kernel mono of* φ and π*,*
- ε *the cokernel of* φ and ι*,*
- π *is the cokernel of* κ*, and*
- ι *is the kernel of* ε*.*

PROOF. Set $I := \operatorname{im}\varphi$, the image of φ, and define π and ι as the image epi and mono, respectively[7]. □

Corollary 2.2.9. *Let $\mathcal{A}$ be an* ABEL*ian category. Then*

- *every mono is the kernel mono of its cokernel epi;*
- *every epi is the cokernel epi of its kernel mono;*

---

[6]It the categorical version of the co-restriction of φ to its image.

[7]Alternatively, set $I := \operatorname{coim}\varphi$ and define π and ι as the co-image epi and mono, respectively. This is equivalent since $\operatorname{coim}\varphi$ and $\operatorname{im}\varphi$ are isomorphic with the explicit isomorphism $\overline{\varphi}$.

In fact, both are equivalent to the defining axiom (Ab).

- *every morphism which is both mono and epi is an isomorphism.*

Example 2.2.10.

(a) (Grps) and (Rngs) are not even pre-additive.
(b) (Ab) is an ABELian category.
(c) More generally, the category R-Mod of left R-modules is an ABELian category. Likewise for the Mod-R, the category of right R-modules.
(d) The full subcategory $R\text{-mod} \subset R\text{-Mod}$ of **finitely generated (f.g.)** left R-modules over a left NOETHERian ring R is an ABELian category. Likewise for $\text{mod-}R \subset \text{Mod-}R$, the category of f.g. right R-modules over a right NOETHERian ring R.
(e) The dual of an ABELian category is again ABELian. An interesting discussion about the opposite category of a module category can be found here [**Ste**].

Exercise 2.3. Show that the full subcategory (tfAb) $\subset$ (Ab) of torsion-free ABELian groups is pre-ABELian but not ABELian. For this consider the map $\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z}$ and prove that it is mono and epi (!) but not an isomorphism *in* (tfAb).

As we have seen in Proposition 1.2.23, the existence of kernels and cokernels implies that of push-outs and pull-backs. They in turn imply the existence of intersections (meets) and sums (joints) in the lattice of subobjects. Dually, there is a lattice of factor objects and, by duality, it has joins (subdirect product) and meets (largest common factor).

In categories with kernels and cokernels there is a GALOIS connection[8] between the lattice of subobjects and the lattice of factor objects: To a subobject $K \overset{\iota}{\hookrightarrow} M$ we can associate the factor object

$$M/K := \operatorname{coker}\left(K \overset{\iota}{\hookrightarrow} M\right)$$

represented by the cokernel epi of ι. And to a factor object $M \overset{\pi}{\twoheadrightarrow} C$ we can associate the subobject $\ker\left(M \overset{\pi}{\twoheadrightarrow} C\right)$ represented by the kernel mono of π. Corollary 2.2.9 now says that in ABELian categories this GALOIS connection is a bijection. Hence, we can draw subobjects and factor objects using just one HASSE diagram. The second isomorphism of NOETHER implies the modularity of these lattices.

Let $L \overset{\tau}{\hookrightarrow} M \overset{\kappa}{\hookleftarrow} K$ be two subobjects of M with $L \leq K$ in some ABELian category. In Exercise 1.1.4 we saw how the unique lift $L \overset{\tau/\kappa}{\hookrightarrow} K$ of τ along the dominant κ realizes L as a subobject of K.

Definition 2.2.11. We then call the factor object

$$K/L := \operatorname{coker}\left(L \overset{\tau/\kappa}{\hookrightarrow} K\right)$$

together with the two morphisms $K/L \hookrightarrow M/L \twoheadleftarrow M$ a **subfactor object of** M.

The two morphisms $K/L \hookrightarrow M/L \twoheadleftarrow M$ is an example of what we will later call a **generalized mono** from K/L to M.

The definition of subfactors is all what we need to define defects of exactness, one of the central notions in homological algebra.

Definition 2.2.12. Let $\mathcal{A}$ be a category, with zero when necessary. Two composable $\mathcal{A}$-morphisms $M \overset{\varphi}{\to} N \overset{\psi}{\to} L$ are called a **short sequence**. We call it a **differential** short sequence if $\varphi\psi = 0$.

If $\mathcal{A}$ is ABELian then we define the **defect of exactness (at** N**)** as the subfactor object of N

$$\operatorname{Def}(\varphi, \psi) = \ker \psi / \operatorname{im} \varphi,$$

[8]monotone or antitone, depending on how we define both.

where the two subobjects $\operatorname{im}\varphi \leq \ker\psi$ of N are represented by the image mono and the kernel mono $\operatorname{im}\varphi \hookrightarrow N \hookleftarrow \ker\psi$, respectively[9]. The sequence is called **exact (at N)** if $\operatorname{Def}(\varphi,\psi) = 0$.

Exercise 2.4. Let $M \xrightarrow{\varphi} N \xrightarrow{\psi} L$ be differential sequence in an ABELian category. The following statements are equivalent:

- The sequence is exact at N.
- The image mono of φ is a kernel mono of ψ.
- The image epi of ψ is cokernel epi of φ.

Definition 2.2.13. Let $\mathcal{A}$ be a category, with zero or ABELian when necessary.

(a) A **sequence** $M_\bullet$ in $\mathcal{A}$ is a (possibly doubly unbounded) *numbered* array consisting of short sequences in $\mathcal{A}$

$$M_\bullet : \cdots \xleftarrow{\partial_{i-1}} M_{i-1} \xleftarrow{\partial_i} M_i \xleftarrow{\partial_{i+1}} M_{i+1} \xleftarrow{\partial_{i+2}} \cdots ,$$

i.e., with $t(\partial_i) = s(\partial_{i-1})$ for all i.
If $M_\bullet$ consists of short differential sequences (i.e., $\partial_{i+1}\partial_i = 0$ for all i) then we call it a **complex**[10]. The defect of exactness

$$H_i(M_\bullet) := \operatorname{Def}(\partial_{i+1},\partial_i) = \ker\partial_i / \operatorname{im}\partial_{i+1}$$

is called the **i-th homology of $M_\bullet$** or the **homology at M_i**. It is a subfactor object of M_i. If the complex is left bounded then its left most homology is the cokernel of the left most morphism. At a right boundary we take the kernel of the right most morphism.

(b) A **co-sequence** $M^\bullet$ in $\mathcal{A}$ is a (possibly doubly unbounded) *numbered* array of short sequences in $\mathcal{A}$

$$M^\bullet : \cdots \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots ,$$

i.e., with $t(d^i) = s(d^{i+1})$ for all i.
If $M^\bullet$ consists of differential sequences (i.e., $d^{i-1}d^i = 0$ for all i) then we call it a **cocomplex**[11]. The defect of exactness

$$H^i(M^\bullet) := \operatorname{Def}(d^{i-1},d^i) = \ker d^i / \operatorname{im} d^{i-1}$$

is called the **i-th cohomology of $M^\bullet$** or the **cohomology at M^i**. It is a subfactor object of M^i. If the cocomplex is left bounded then its left most homology is the kernel of the left most morphism. At a right boundary we take the cokernel of the right most morphism.

The only difference between complexes and cocomplexes is whether i decreases or increases in the direction of the arrows. Of course, this can only has any relevance if the numbers i carry a mathematical meaning, e.g., being dimensions or codimensions in some context.

Before we see where complexes and cocomplexes naturally arise we study categories of modules which are *constructively* ABELian. Only then will we be able to compute the (co)homologies once we encounter complexes. Rest lecture 11

We call a (co)complex **acyclic** if its (co)homology at each intermediate[12] object vanishes. The notions of **left** resp. **right acyclicity (or exactness)** are evident. Finally, we call it **exact** if it is both left and right acyclic[13].

In categories with zero we can always think of complexes as doubly infinite ones by augmenting zero morphisms on both sides. However, this might alter the meaning of acyclicity!

Definition 2.2.14. A **chain morphism (in $\mathcal{A}$)** $\mu_\bullet : M_\bullet \to N_\bullet$ between two complexes $M_\bullet$ and $N_\bullet$ in $\mathcal{A}$ is an array of morphisms $\mu_i : M_i \to N_i$ such that following diagram commutes:

[9] The latter dominates the former by the universal property of the kernel.
[10] or **chain complex** or **homological complex**.
[11] or **cochain complex** or **cohomological complex**.
[12] i.e., an object which is a source and a target.
[13] Most people use acyclic for what is called here exact.

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & M_{i-1} & \longleftarrow & M_i & \longleftarrow & M_{i+1} & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mu_{i-1}} & & \downarrow{\scriptstyle \mu_i} & & \downarrow{\scriptstyle \mu_{i+1}} & & \\
\cdots & \longleftarrow & N_{i-1} & \longleftarrow & N_i & \longleftarrow & N_{i+1} & \longleftarrow & \cdots
\end{array}$$

The dual notion is that of a **cochain morphism**.

Chain complexes and their chain morphisms form a category which we will denote by $\mathrm{Ch}_\bullet \mathcal{A}$. The full subcategories of all complexes with $M_i = 0$ for $i \geq 0$, $i \ll 0$, $i \leq 0$, $i \gg 0$, $|i| \gg 0$ are denoted by $\mathrm{Ch}_{\geq 0}$, $\mathrm{Ch}_+ \mathcal{A}$, $\mathrm{Ch}_{\leq 0}$, $\mathrm{Ch}_- \mathcal{A}$, $\mathrm{Ch}_\mathrm{b} \mathcal{A}$, respectively. For cochain complexes we write upper indices $\mathrm{Ch}^\bullet \mathcal{A}$, etc.

Definition 2.2.15. We call a subobject[14] of $M_\bullet$ a **subcomplex**. Dually, a **factor complex** is a factor object of $M_\bullet$.

REMARK 2.2.16. Let $\mathcal{A}$ be additive or ABELian category. Then the categories $\mathrm{Ch}_\bullet \mathcal{A}$, $\mathrm{Ch}_{\geq 0} \mathcal{A}$, $\mathrm{Ch}_+ \mathcal{A}$, $\mathrm{Ch}_{\leq 0} \mathcal{A}$, $\mathrm{Ch}_- \mathcal{A}$, $\mathrm{Ch}_\mathrm{b} \mathcal{A}$ are again additive or ABELian, respectively.

If $\mathcal{A}$ is ABELian then a chain morphism $\mu_\bullet$ is mono[15] (resp. epi, resp. iso) iff μ_i is mono (resp. epi, resp. iso) for all $i \in \mathbb{Z}$.

Let $M_\bullet = (M_\bullet, \partial_\bullet^M)$ be a complex in the additive category $\mathcal{A}$. A **degree** n chain morphism $M_\bullet \to N_\bullet$ is a chain morphism $M_\bullet \to N_{\bullet+n}$ or, equivalently, $M_{\bullet-n} \to N_\bullet$.

For example, the array $\partial_\bullet^M$ itself can be interpreted as a degree -1 chain endomorphism of $M_\bullet$, i.e., as a chain morphism $M_\bullet \to M_{\bullet-1}$.

In this language a chain morphism $\mu_\bullet : M_\bullet \to N_\bullet$ satisfies $\partial_\bullet^M \mu_\bullet = \mu_\bullet \partial_\bullet^N$ as chain morphisms $M_\bullet \to N_{\bullet-1}$ (or $M_{\bullet+1} \to N_\bullet$.

Definition 2.2.17. Let $M_\bullet$ be a complex in an ABELian category. The image of $M_{\bullet+1} \to M_\bullet$ is called the subcomplex of **boundaries** and denoted by $B(M_\bullet)$. The kernel of $M_\bullet \to M_{\bullet-1}$ is called the subcomplex of **cycles** and denoted by $Z(M_\bullet)$. It follows that $B(M_\bullet) \leq Z(M_\bullet)$ and $H_i(M_\bullet) = Z(M_i)/B(M_i)$.

[14] in any of the above categories $\mathrm{Ch}_? \mathcal{A}$

[15] in any of the above categories $\mathrm{Ch}_? \mathcal{A}$

CHAPTER 3

Categories of finite presentations

We could've started developing advanced abstract algorithms in ABELian categories but it makes more sense to first describe a basic class of categories of finitely presented modules, which are not only ABELian but *constructively* ABELian [**BLH11**]. Once we have such a constructively ABELian category $\mathcal{A}$, i.e., once we have for $\mathcal{A}$ specified algorithms for all the existential quantifiers and disjunctions indicated by $\boxed{\text{boxes}}$ in the previous chapters, then all abstract algorithms become algorithms in the usual sense. In fact, the boxes describe all the needed interfaces between the abstract and the concrete part of a well-structured computer implementation. The above mentioned class of finitely presented module categories has already a wide range of applications. However, by modeling any other ABELian category $\mathcal{A}$ (e.g., of coherent sheaves on certain varieties) on a constructively ABELian one we get a proof that $\mathcal{A}$ itself is constructively ABELian, where all abstract algorithms which we will develop later for ABELian categories will be automatically applicable.

3.1. Computable rings

In order to describe this basic class of categories of finitely presented modules which are constructively ABELian we first need to specify the underlying rings. Recall that all rings we consider are unitial. Let R be such a ring. All matrices over R will finite number of rows and columns. Empty matrices with no rows or no columns are explicitly allowed.

Definition 3.1.1.

- A **constructive set** S consists of two algorithms, one to *decide membership* in S (for *any* given input) and another to *decide equality* of elements in S.
- A **constructive ABELian group** H (additively written) is a constructive set together with an explicit element $0 \in H$ and algorithms for addition and subtraction satisfying the usual axioms.
- A **constructive ring** R is a constructive ABELian group together with an explicit $1 \in R$ and an algorithm for multiplication satisfying the usual axioms.

REMARK 3.1.2.

- We did not require to have an algorithm which decides if a constructive set is empty or not.
- Likewise, we did not require to have an algorithm which decides if a constructive ABELian group is trivial or not.
- Since we can decide $0 = 1$ in a constructive ring R we can decide whether or not R is the zero ring.

We now introduce a terminology analogous to a one we used for morphisms. End lecture 5

Definition 3.1.3. An $r \times c$ matrix $\mathtt{A}$ over R is said to

(row) **row-dominate** $\mathtt{B} \in R^{r' \times c}$ if there $\boxed{\text{exists}}$ a matrix $\mathtt{X} \in R^{r' \times r}$ such that $\mathtt{B} = \mathtt{XA}$ (we write $\mathtt{A} \geq_{\text{row}} \mathtt{B}$);

(col) **column-dominate** $\mathtt{B} \in R^{r \times c'}$ if there $\boxed{\text{exists}}$ a matrix $\mathtt{X} \in R^{c \times c'}$ such that $\mathtt{B} = \mathtt{AX}$ (we write $\mathtt{A} \geq_{\text{col}} \mathtt{B}$).

Exercise 3.1. Show that $\mathtt{A} =_{\text{row}} \mathtt{A}' :\iff \mathtt{A} \geq_{\text{row}} \mathtt{A}'$ and $\mathtt{A}' \geq_{\text{row}} \mathtt{A}$ is an equivalence relation[1]. The analogous statement holds for $=_{\text{col}}$ and $\geq_{\text{col}}$.

The syzygy matrices will be the essential ingredient of kernels and their kernel monos.

[1]Both matrices generate the same **row-space**

Definition 3.1.4. Let $\mathtt{A} \in R^{r\times c}$ be a matrix over R.

(row) A matrix $\mathtt{S} \in R^{a\times r}$ is called a matrix of **row syzygies of** $\mathtt{A}$ if
- $\mathtt{SA} = 0$;
- $\mathtt{S}$ row-dominates any other matrix $\mathtt{S}' \in R^{a'\times r}$ with $\mathtt{S}'\mathtt{A} = 0$.

(col) A matrix $\mathtt{S} \in R^{c\times a}$ is called a matrix of **column syzygies of** $\mathtt{A}$ if
- $\mathtt{AS} = 0$;
- $\mathtt{S}$ column-dominates any other matrix $\mathtt{S}' \in R^{c\times a'}$ with $\mathtt{AS}' = 0$.

Definition 3.1.5. A **left computable ring** is a constructive ring R together with algorithms for solving left sided equations $\mathtt{B} = \mathtt{XA}$ (for given matrices $\mathtt{A}$ and $\mathtt{B}$ over R with equal number of columns), i.e., with three algorithms

- $\mathtt{DecideZeroRows}(\mathtt{B},\mathtt{A})$ which decides if $\mathtt{A} \geq_{\text{row}} \mathtt{B}$, i.e., if there exists a solution matrix $\mathtt{X}$ with $\mathtt{XA} = \mathtt{B}$: the algorithm returns a matrix $\mathtt{B}'$ (having the same shape as $\mathtt{B}$) for which the equation $\mathtt{XA} = \mathtt{B} - \mathtt{B}'$ is solvable and where the i-th row $\mathtt{B}'_i$ is zero iff the equation $\mathtt{xA} = \mathtt{B}_i$ is solvable.[2] In particular, the equation $\mathtt{XA} = \mathtt{B}$ is solvable iff $\mathtt{DecideZeroRows}(\mathtt{B},\mathtt{A}) = 0$;
- $\mathtt{DecideZeroRowsEffectively}(\mathtt{B},\mathtt{A})$ computes a matrix $\mathtt{T}$ satisfying $\mathtt{B} + \mathtt{TA} = \mathtt{B}'$, where $\mathtt{B}' = \mathtt{DecideZeroRows}(\mathtt{B},\mathtt{A})$. In particular, if the equation $\mathtt{XA} = \mathtt{B}$ is solvable then

$$\mathtt{X} := -\mathtt{T} =: \mathtt{RightDivide}(\mathtt{B},\mathtt{A});$$

- $\mathtt{SyzygiesOfRows}(\mathtt{A})$ which computes a matrix $\mathtt{S}$ of row syzygies of $\mathtt{A}$.

Analogously define a **right computable ring** with its three algorithms for solving right sided equations $\mathtt{AX} = \mathtt{B}$

- $\mathtt{DecideZeroColumns}(\mathtt{B},\mathtt{A})$;
- $\mathtt{DecideZeroColumnsEffectively}(\mathtt{B},\mathtt{A})$ and in the affirmative case

$$\mathtt{X} := \mathtt{LeftDivide}(\mathtt{A},\mathtt{B});$$

- $\mathtt{SyzygiesOfColumns}(\mathtt{A})$.

Finally, we call a ring **computable** if it is both left and right computable.

Exercise 3.2. An **involution** on a ring R is an anti-isomorphism $\theta : R \to R$ with $\theta^2 = \mathrm{id}_R$, i.e., θ is an isomorphism of the underlying ABELian group $(R, +)$ and $\theta(1) = 1$, $\theta(\theta(a)) = a$, and $\theta(ab) = \theta(b)\theta(a)$ for all $a, b \in R$. Construct an involution on the following rings.

(a) R is a commutative ring.
(b) R is the WEYL algebra $k[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$.

Exercise 3.3. The following are equivalent for any ring R with an involution:

- R is left computable.
- R is right computable.
- R is computable.

Exercise 3.4. Let R be a commutative computable ring and $I = \langle a_1, \ldots, a_n\rangle \trianglelefteq R$ an (explicitly) finitely generated ideal of R. Prove that the residue class ring R/I is again computable.

REMARK 3.1.6. The following rings are (left) computable:

(a) A constructive field k equipped with the GAUSSian normal form algorithm, more precisely, an algorithm to compute the row reduced echelon form (RREF).
(b) A EUCLIDian ring R equipped with a HERMITE normal form algorithm, e.g., $R = \mathbb{Z}$ or $R = k[x]$.
(c) A polynomial ring $R = k[x_1, \ldots, x_n]$ over a computable field k equipped with an algorithm to compute reduced GRÖBNER bases (of rows of rectangular matrices).
(d) In fact any ring R with a GRÖBNER basis notion and equipped with an algorithm to compute reduced GRÖBNER bases (of rows of rectangular matrices), e.g., $R = k[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$.

[2] So we do not require a "normal form", but only a mechanism to decide if a row is zero modulo some relations.

PROOF. Apply such an algorithm to the block matrix[3] $\left(\begin{smallmatrix} 1 & \mathtt{B} & 0 \\ 0 & \mathtt{A} & 1 \end{smallmatrix}\right)$ and obtain $\left(\begin{smallmatrix} 1 & \mathtt{B}' & -\mathtt{X} \\ 0 & \mathtt{A}' & \mathtt{Y} \\ 0 & 0 & \mathtt{S} \end{smallmatrix}\right)$. The last matrix is obviously the product $\left(\begin{smallmatrix} 1 & -\mathtt{X} \\ 0 & \mathtt{Y} \\ 0 & \mathtt{S} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \mathtt{B} & 0 \\ 0 & \mathtt{A} & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \mathtt{B}' & -\mathtt{X} \\ 0 & \mathtt{A}' & \mathtt{Y} \\ 0 & 0 & \mathtt{S} \end{smallmatrix}\right)$; the simple form of the first column $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ of the transformation matrix asserts that the normal form algorithm does not auto-transform the first row block $\left(\begin{smallmatrix} 1 & \mathtt{B} & 0 \end{smallmatrix}\right)$ of the input matrix as its left most block is an identity matrix $\mathtt{1}$. Note that $\mathtt{B}' = \mathtt{B} - \mathtt{XA}$ and the left sided equation $\mathtt{XA} = \mathtt{B}$ is solvable (with solution $\mathtt{X}$) iff $\mathtt{B}'$ vanishes, yielding the algorithms $\mathtt{B}' = \mathtt{DecideZeroRows}(\mathtt{B}, \mathtt{A})$ and $(\mathtt{B}', -\mathtt{X}) = \mathtt{DecideZeroRowsEffectively}(\mathtt{B}, \mathtt{A})$. $\mathtt{S}$ is a matrix of row syzygies, yielding the algorithm $\mathtt{S} = \mathtt{SyzygiesOfRows}(\mathtt{A})$. □

We write $\mathtt{A}' = \mathtt{YA} = \mathtt{BasisOfRows}(\mathtt{A})$ and $(\mathtt{A}', \mathtt{Y}) = \mathtt{BasisOfRowsCoeff}(\mathtt{A})$.

REMARK 3.1.7. It is important to emphasize that we do need `DecideZeroRows` to compute any kind of "normal form", unless the normal form is zero. This is reflected in our choice of the name `DecideZeroRows`.

REMARK 3.1.8. In fact, the polynomial ring $k[x_1, x_2, \ldots]$ in infinitely many indeterminates is computable! So computability does not imply NOETHERianity (left or right) and the former is in our context the correct substitute for the latter.

3.2. A constructively ABELian category of matrices

Now we define a category of matrices which will serve as a "model" for the category of finitely presented modules. This category will turn out to be constructively ABELian when R ist left (resp. right) computable.

Definition 3.2.1. Let R be a ring. Define the **category** R-fpres **of (finite) left presentations over** R as follows:

(Obj) $(R\text{-fpres})_0$:= the set of all matrices $\mathtt{M}$ over R (including the empty one) we call the number of columns of a matrix $\mathtt{M}$ the **number of generators**[4] and denote it by $g_{\mathtt{M}}$;

(Mor) $(R\text{-fpres})_1$:= the set of **left compatible triples** $(\mathtt{M}, \mathtt{A}, \mathtt{N})$ over R *modulo* a certain equivalence relation $\sim$, where a triple of matrices $(\mathtt{M}, \mathtt{A}, \mathtt{N})$ over R is called left compatible if
- $\mathtt{A}$ is a $g_{\mathtt{M}} \times g_{\mathtt{N}}$ matrix[5];
- $\mathtt{N} \geq_{\text{row}} \mathtt{MA}$, i.e., the left sided equation $\mathtt{X_A N} = \mathtt{MA}$ is solvable[6] for some matrix $\mathtt{X}$.

The equivalence relation $\sim$ is defined as follows: $(\mathtt{M}, \mathtt{A}, \mathtt{N}) \sim (\mathtt{M}', \mathtt{A}', \mathtt{N}')$ if
- $\mathtt{M} = \mathtt{M}'$;
- $\mathtt{N} = \mathtt{N}'$;
- $\mathtt{N} \geq_{\text{row}} \mathtt{A} - \mathtt{A}'$, i.e., there exists[7] a matrix $\mathtt{Y}$ such that[8] $\mathtt{A} - \mathtt{A}' = \mathtt{YN}$.

The identity, source, target, and composition defined as follows:

(1) $1_{\mathtt{M}} := \mathtt{IdentityMorphism}(\mathtt{M}) := (\mathtt{M}, \mathtt{1}, \mathtt{M})$, where $\mathtt{1}$ is the identity matrix over R with dimension equal to the number of columns of the matrix $\mathtt{M}$.

(s) $s((\mathtt{M}, \mathtt{A}, \mathtt{N})) := \mathtt{Source}((\mathtt{M}, \mathtt{A}, \mathtt{N})) := \mathtt{M}$;

(t) $t((\mathtt{M}, \mathtt{A}, \mathtt{N})) := \mathtt{Target}((\mathtt{M}, \mathtt{A}, \mathtt{N})) := \mathtt{N}$;

(μ) $\mu((\mathtt{M}, \mathtt{A}, \mathtt{N}), (\mathtt{N}, \mathtt{B}, \mathtt{L})) = \mathtt{Compose}((\mathtt{M}, \mathtt{A}, \mathtt{N}), (\mathtt{N}, \mathtt{B}, \mathtt{L})) = (\mathtt{M}, \mathtt{A}, \mathtt{N})(\mathtt{N}, \mathtt{B}, \mathtt{L}) = (\mathtt{M}, \mathtt{AB}, \mathtt{L})$, where $\mathtt{AB}$ is the matrix multiplication.

We occasionally denote a morphism $(\mathtt{M}, \mathtt{A}, \mathtt{N})$ by $\mathtt{M} \xrightarrow{\mathtt{A}} \mathtt{N}$.
The **category** fpres-R **of (finite) left presentations over** R is defined analogously. Everything we do below also holds for fpres-R.

[3] In practice one would avoid the construction of such big partially sparse partially dense block matrices and writes specific variants to construct $\mathtt{X}$, $\mathtt{Y}$ and $\mathtt{S}$ directly form $\mathtt{A}$ and $\mathtt{B}$.

[4] We will justify this name in the next chapter.

[5] in particular, the matrix multiplication $\mathtt{MA}$ is defined.

[6] This is part of the membership decision algorithm in $(R\text{-fpres})_1$.

[7] This is part of the membership equality algorithm in $(R\text{-fpres})_1$.

[8] We say that the matrix $\mathtt{A}$ is unique up to a multiple of $\mathtt{N}$.

Indeed the composed tripple $(\mathtt{M}, \mathtt{AB}, \mathtt{N})$ is again compatible: There exists an $\mathtt{X_A}$ such that $\mathtt{X_A N} = \mathtt{MA}$ and $\mathtt{X_B}$ such that $\mathtt{X_B L} = \mathtt{NB}$ and therefore $\mathtt{M(AB)} = \mathtt{(MA)B} = \mathtt{(X_A N)B} = \mathtt{X_A(NB)} = \mathtt{X_A(X_B L)} = \mathtt{(X_A X_B)L}$. Furthermore, the composition is well-defined for classes: if $(\mathtt{M}, \mathtt{A}, \mathtt{N}) \sim (\mathtt{M}, \mathtt{A}', \mathtt{N})$ via $\mathtt{Y_{A,A'}}$ and $(\mathtt{N}, \mathtt{B}, \mathtt{L}) \sim (\mathtt{N}, \mathtt{B}', \mathtt{L})$ via $\mathtt{Y_{B,B'}}$ then $\mathtt{AB} - \mathtt{A'B'} = (\mathtt{A}' + \mathtt{Y_{A,A'}N})(\mathtt{B}' + \mathtt{Y_{B,B'}L}) - \mathtt{A'B'} = \mathtt{Y_{A,A'}} \underbrace{\mathtt{NB'}}_{=\mathtt{X_{B'}L}} + \mathtt{A'Y_{B,B'}L} + \mathtt{Y_{A,A'}NY_{B,B'}L} = (\ldots)\mathtt{L}$. The remaining defining properties are all obvious.

When R is left (resp. right) computable then R-fpres (resp. fpres-R) is a constructive category in the following sense:

Definition 3.2.2. We call a category $\mathcal{A}$ **constructive** if $\mathcal{A}_0$, $\mathcal{A}_1$ are constructive sets and all four structure maps $1, s, t, \mu$ are realized by algorithms.

REMARK 3.2.3. More generally, we call a mathematical structure **constructive** if all existential quantifiers and all disjunctions appearing in its defining axioms are realized by algorithms. All the above constructivity notions were in this sense special cases. However, as this notion is usually not treated in classical math curricula we decided to define some special cases explicitly to underline the various and maybe unexpected places where such quantifiers occur.

Note that we do *not* require to have an algorithm which decides if two different objects are isomorphic or not. However, in a constructively ABELian category $\mathcal{A}$ the discussion below will provide an algorithm capable of deciding whether a given $\mathcal{A}$-morphism $\alpha : M \to N$ is an isomorphism or not.

Proposition 3.2.4. *If R is left computable then R-fpres is a constructive category.*

PROOF.

- The membership and equality algorithms for $(R\text{-fpres})_0$ are derived from those of R.
- For the membership and equality algorithms for $(R\text{-fpres})_1$ we use the left computability of R.
- For `IdentityMorphism` we use a matrix constructor `IdentityMat` for (possibly 0×0) identity matrices.
- The algorithms for `Source` and `Target` are obvious.
- For `Compose` we use an algorithm `MulMat` for matrix multiplication.

Both algorithms `IdentityMat` and `MulMat` rely on the constructivity of R. □

Our next goal is to prove that R-fpres is in fact constructively ABELian. For the rest of this section let R be a left computable ring.

We start with the pre-additive structure:

Definition 3.2.5. Define the constructive pre-additive structure on R-fpres by the following matrix algorithms

(+) $(\mathtt{M}, \mathtt{A}, \mathtt{N}) + (\mathtt{M}, \mathtt{B}, \mathtt{N}) := (\mathtt{M}, \mathtt{A} + \mathtt{B}, \mathtt{N})$;
(−) $(\mathtt{M}, \mathtt{A}, \mathtt{N}) - (\mathtt{M}, \mathtt{B}, \mathtt{N}) := (\mathtt{M}, \mathtt{A} - \mathtt{B}, \mathtt{N})$;
(0) The zero morphism is given by $\mathtt{ZeroMorphism}(\mathtt{M}, \mathtt{N}) := (\mathtt{M}, \mathtt{0}, \mathtt{N})$.

We call the corresponding matrix algorithms `AddMat`, `SubMat`, and a constructor `ZeroMat` for (possibly empty) zero matrices, respectively. All of them rely on R merely being a constructive ring.

This constructive pre-additive structure is, as expected, uniquely determined and hence reproducible from the constructive additivity of the category discussed below.

REMARK 3.2.6. If R is left computable then R-fpres is a constructive category with zero.

PROOF. The 0×0 matrix $\mathtt{0}$ over R is a zero object in R-fpres. Indeed, there is a unique morphism $(\mathtt{M}, \mathtt{0}, \mathtt{0})$ and $(\mathtt{0}, \mathtt{0}, \mathtt{N})$ with middle matrices being of the shape $? \times 0$ and $0 \times ?$, respectively. All involved empty matrices can be constructed using the matrix constructor `ZeroMat`. □

As expected, the zero morphism $0_{\mathtt{MN}} := \mathtt{M} \to \mathtt{0} \to \mathtt{N} = (\mathtt{M}, \mathtt{0}, \mathtt{0})(\mathtt{0}, \mathtt{0}, \mathtt{N})$ coincides with the zero morphism $(\mathtt{M}, \mathtt{0}, \mathtt{N})$, where the zero matrix is the product of the two empty matrices.

End lecture 6

Proposition 3.2.7. *If R is left computable then R-fpres is constructively additive.*

PROOF. Let $\mathtt{M}, \mathtt{N}$ be two objects in R-fpres. Consider the object

$$\mathtt{M} \oplus \mathtt{N} := \mathtt{ProductObject}(\mathtt{M}, \mathtt{N}) := \mathtt{DiagMat}(\mathtt{M}, \mathtt{N})^{9} := \begin{pmatrix} \mathtt{M} & \mathtt{0} \\ \mathtt{0} & \mathtt{N} \end{pmatrix}$$

together with the two morphisms (given by the two compatible triples)

$$\pi_{\mathtt{M}} := \mathtt{ProjectionOnLeftFactor}(\mathtt{M}, \mathtt{N}) := \left(\mathtt{M} \oplus \mathtt{N}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathtt{M}\right),$$
$$\pi_{\mathtt{N}} := \mathtt{ProjectionOnRightFactor}(\mathtt{M}, \mathtt{N}) := \left(\mathtt{M} \oplus \mathtt{N}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathtt{N}\right)$$

as a candidate for the product of $\mathtt{M}$ and $\mathtt{N}$ and its two projections. This relies on a matrix constructor algorithms DiagMat and StackMat to stack matrices. Indeed for two arbitrary morphisms $\varphi : \mathtt{L} \xrightarrow{\mathtt{A}} \mathtt{M}$ and $\psi : \mathtt{L} \xrightarrow{\mathtt{B}} \mathtt{N}$ there exactly one morphism (i.e., one equivalence class of compatible triples)

$$\{\varphi, \psi\} := \mathtt{ProductMorphism}(\varphi, \psi) := \left(\mathtt{L}, \begin{pmatrix} \mathtt{A} & \mathtt{B} \end{pmatrix}, \mathtt{M} \oplus \mathtt{N}\right),$$

such that $\{\varphi, \psi\}\pi_{\mathtt{M}} = \varphi$ and $\{\varphi, \psi\}\pi_{\mathtt{N}} = \psi$, as can be verified by block matrix multiplication. For this we use a constructor AugmentMat to augment matrices. ☐

REMARK 3.2.8. Constructing the embeddings in the biproduct and the coproduct morphism now follows from Remark 2.2.4. The following algorithms are in this sense derived algorithms:

- $\iota_{\mathtt{M}} := \mathtt{EmbeddingOfLeftCofactor}(\mathtt{M}, \mathtt{N}) := \{1_{\mathtt{M}}, 0_{\mathtt{MN}}\} = \left(\mathtt{M}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \mathtt{M} \oplus \mathtt{N}\right)$.
- $\iota_{\mathtt{N}} := \mathtt{EmbeddingOfRightCofactor}(\mathtt{M}, \mathtt{N}) := \{0_{\mathtt{NM}}, 1_{\mathtt{N}}\} = \left(\mathtt{N}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \mathtt{M} \oplus \mathtt{N}\right)$.
- For two morphisms $\varphi : \mathtt{M} \xrightarrow{\mathtt{A}} \mathtt{L}$ and $\psi : \mathtt{N} \xrightarrow{\mathtt{B}} \mathtt{L}$

  $$\langle \varphi, \psi \rangle := \mathtt{CoproductMorphism}(\varphi, \psi) := \pi_{\mathtt{M}}\varphi + \pi_{\mathtt{N}}\psi = \left(\mathtt{M} \oplus \mathtt{N}, \begin{pmatrix} \mathtt{A} \\ \mathtt{B} \end{pmatrix}, \mathtt{L}\right).$$

- As mentioned above, the sum $\varphi + \psi : \mathtt{M} \xrightarrow{\mathtt{A+B}} \mathtt{N}$ of two parallel morphisms $\varphi : \mathtt{M} \xrightarrow{\mathtt{A}} \mathtt{N}$, $\psi : \mathtt{M} \xrightarrow{\mathtt{B}} \mathtt{N}$ can be reconstructed as

  $$\varphi + \psi := \mathtt{Add}(\varphi, \psi) := \{1_{\mathtt{M}}, 1_{\mathtt{M}}\}\langle \varphi, \psi \rangle = \left(\mathtt{M}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathtt{M} \oplus \mathtt{M}\right)\left(\mathtt{M} \oplus \mathtt{M}, \begin{pmatrix} \mathtt{A} \\ \mathtt{B} \end{pmatrix}, \mathtt{N}\right).$$

Before we discuss the pre-ABELian structure we need a relative version of the two matrix algorithms RightDivide and SyzygiesOfRows which we can deduce as derived algorithms. We start with the relative version of domination.

Definition 3.2.9. Let $\mathtt{A}, \mathtt{B}, \mathtt{N}$ be stackable[10] matrices over R. The matrix $\mathtt{A}$ is said to **row-dominate** $\mathtt{B}$ **modulo** $\mathtt{N}$ if there exists matrices $\mathtt{X}, \mathtt{Y}$ such that $\mathtt{B} = \mathtt{XA} + \mathtt{YN}$. The definition of relative column-domination is analogous.

Since $\mathtt{B} = \mathtt{XA} + \mathtt{YN} = \begin{pmatrix} \mathtt{X} & \mathtt{Y} \end{pmatrix} \begin{pmatrix} \mathtt{A} \\ \mathtt{N} \end{pmatrix}$ we can derive $\mathtt{X}$ (and $\mathtt{Y}$) form RightDivide and StackMat

$$\begin{pmatrix} \mathtt{X} & \mathtt{Y} \end{pmatrix} := \mathtt{RightDivide}\left(\mathtt{B}, \begin{pmatrix} \mathtt{A} \\ \mathtt{N} \end{pmatrix}\right).$$

We define the relative version[11] of RightDivide by applying CertainColumns to extract $\mathtt{X}$

$$\mathtt{X} =: \mathtt{RightDivide}(\mathtt{B}, \mathtt{A}, \mathtt{N}).$$

Analogously we derive the relative version $\mathtt{LeftDivide}(\mathtt{A}, \mathtt{B}, \mathtt{N})$.

Definition 3.2.10. Let $\mathtt{A}$ and $\mathtt{N}$ be two stackable matrices over R. We say that $\mathtt{K}$ is a matrix of **row syzygies of** $\mathtt{A}$ **modulo** $\mathtt{N}$ if

- $\mathtt{N} \geq_{\text{row}} \mathtt{KA}$;
- $\mathtt{K}$ row-dominates any such matrix $\mathtt{T}$ (i.e., any $\mathtt{T}$ where $\mathtt{N} \geq_{\text{row}} \mathtt{TA}$).

For the stacked matrix $\begin{pmatrix} \mathtt{A} \\ \mathtt{N} \end{pmatrix}$ we write $\mathtt{SyzygiesOfRows}\left(\begin{pmatrix} \mathtt{A} \\ \mathtt{N} \end{pmatrix}\right) = \begin{pmatrix} \mathtt{K} & \mathtt{L} \end{pmatrix}$ with $\mathtt{KA} + \mathtt{LN} = 0$ and define[12]

$$\mathtt{SyzygiesOfRows}(\mathtt{A}, \mathtt{N}) := \mathtt{K},$$

for which we need a matrix algorithm CertainColumns to extract $\mathtt{K}$.

---

[9] Of course, DiagMat can be derived from the constructors ZeroMat, StackMat, and AugmentMat.

[10] i.e., with the same number of columns.

[11] In practice, one can derive more efficient algorithms to compute the relative version of RightDivide.

[12] Again, one can derive more efficient algorithms to compute the relative version of SyzygiesOfRows.

Proposition 3.2.11. *If R is left computable then R-fpres is constructively pre-*ABEL*ian.*

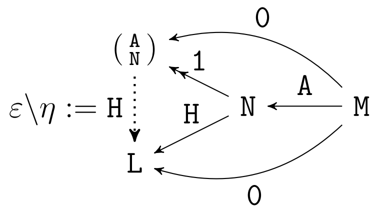
PROOF. Let $\varphi : \mathtt{M} \xrightarrow{\mathtt{A}} \mathtt{N}$ be a morphism in R-fpres. We start with the existence of cokernels. Consider the object

$$\operatorname{coker}\varphi := \mathtt{Cokernel}(\varphi) := \left(\begin{smallmatrix}\mathtt{A}\\ \mathtt{N}\end{smallmatrix}\right)$$

together with the morphism

$$\varepsilon := \mathtt{CokernelEpi}(\varphi) := (\mathtt{N}, \mathtt{1}, \operatorname{coker}\varphi)$$

as a candidate for the cokernel of φ and its cokernel epi.



Indeed, $\varphi\varepsilon = 0$ (since $\left(\begin{smallmatrix}\mathtt{A}\\ \mathtt{N}\end{smallmatrix}\right)$ tautologically row-dominates $\mathtt{A} \cdot \mathtt{1} = \mathtt{A}$) and for any morphism $\eta : \mathtt{N} \xrightarrow{\mathtt{H}} \mathtt{L}$ with $\varphi\eta = 0$ (i.e., $\mathtt{L} \geq_{\text{row}} \mathtt{AH}$) there is exactly one morphism (i.e., one equivalence class of compatible triples)

$$\varepsilon\backslash\eta := \mathtt{Colift}(\eta, \varepsilon) := (\operatorname{coker}\varphi, \mathtt{H}, \mathtt{L}),$$

such that $\varepsilon(\varepsilon\backslash\eta) = \eta$.
To establish the existence of kernels we first define the matrix

$$\mathtt{K} := \mathtt{SyzygiesOfRows}(\mathtt{A}, \mathtt{N}).$$

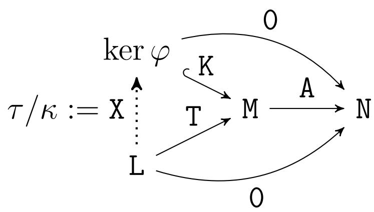
Now consider the object

$$\ker\varphi := \mathtt{Kernel}(\varphi) := \mathtt{SyzygiesOfRows}(\mathtt{K}, \mathtt{M})$$

together with the morphism

$$\kappa := \mathtt{KernelMono}(\varphi) := (\ker\varphi, \mathtt{K}, \mathtt{M})$$

as a candidate for the kernel of φ and its kernel mono.



Indeed, $\kappa\varphi = 0$ (since $\mathtt{N} \geq_{\text{row}} \mathtt{KA}$, by definition of $\mathtt{K}$) and for any morphism $\tau : \mathtt{L} \xrightarrow{\mathtt{T}} \mathtt{M}$ with $\tau\varphi = 0$ (i.e., $\mathtt{N} \geq_{\text{row}} \mathtt{TA}$) we need to prove that there is exactly one morphism

$$\tau/\kappa := \mathtt{Lift}(\tau, \kappa) := (\mathtt{L}, \mathtt{X}, \ker\varphi)$$

such that $(\tau/\kappa)\kappa = \tau$: This last condition means that $\mathtt{M}$ must row-dominate[13] $\mathtt{XK} - \mathtt{T}$. But in fact we know that $\mathtt{K} \geq_{\text{row}} \mathtt{T}$ and we can achieve $\mathtt{XK} - \mathtt{T} = \mathtt{0}$ by setting[14]

$$\mathtt{X} := \mathtt{RightDivide}(\mathtt{T}, \mathtt{K}).$$

With this choice of $\mathtt{X}$ we want to see that $(\mathtt{L}, \mathtt{X}, \ker\varphi)$ is a compatible triple. First note that $\mathtt{M} \geq_{\text{row}} \mathtt{LT} = \mathtt{L}(\mathtt{XK}) = (\mathtt{LX})\mathtt{K}$ since τ is a morphism, i.e., $\mathtt{LX}$ is a matrix of syzygies of $\mathtt{K}$ modulo $\mathtt{M}$. But then, by definition, $\mathtt{SyzygiesOfRows}(\mathtt{K}, \mathtt{M}) = \ker\varphi \geq_{\text{row}} \mathtt{LX}$ and we are done. ☐

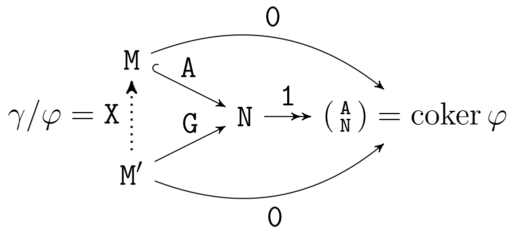
It now remains to prove that R-fpres is ABELian.

Theorem 3.2.12. *If R is left computable then R-fpres is constructively* ABEL*ian.*

[13] or, equivalently, that $\mathtt{K} \geq_{\text{row}} \mathtt{T}$ modulo $\mathtt{M}$.

[14] We also could have used $\mathtt{X} := \mathtt{RightDivide}(\mathtt{T}, \mathtt{K}, \mathtt{M})$. It is not clear to me if this is generically slower.

PROOF. First we will show that any mono is the kernel of its cokernel (cf. Corollary 2.2.9). Let $\varphi : \mathtt{M} \overset{\mathtt{A}}{\hookrightarrow} \mathtt{N}$ be a mono. From the proof of Proposition 3.2.11 we know that $\varepsilon = (\mathtt{N}, \mathtt{1}, \left(\begin{smallmatrix}\mathtt{A}\\\mathtt{N}\end{smallmatrix}\right) = \operatorname{coker}\varphi)$ is the cokernel epi of φ.

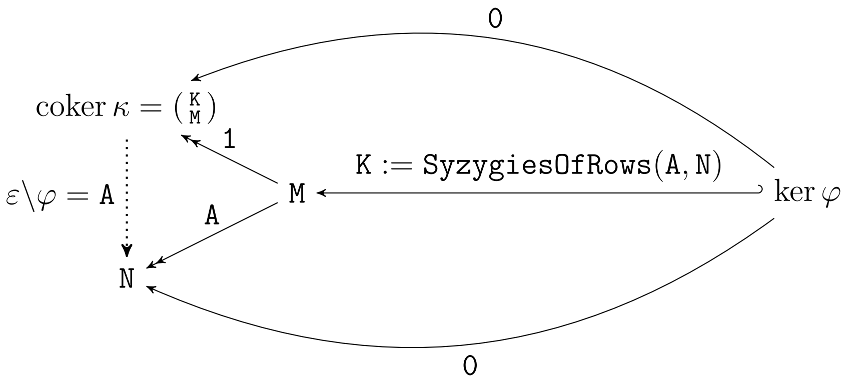


Let $\gamma : \mathtt{M}' \overset{\mathtt{G}}{\to} \mathtt{N}$ be another morphism with $\gamma\varepsilon = 0$, i.e., $\left(\begin{smallmatrix}\mathtt{A}\\\mathtt{N}\end{smallmatrix}\right) \geq_{\text{row}} \mathtt{G} = \mathtt{G}\cdot\mathtt{1}$, or, equivalently, $\mathtt{A} \geq_{\text{row}} \mathtt{G}$ modulo $\mathtt{N}$, i.e., there exists $\mathtt{X}, \mathtt{Y}$ such that $\mathtt{G} = \mathtt{XA} + \mathtt{YN}$. A candidate for the unique lift[15] of γ along φ would be $\gamma/\varphi : \mathtt{M}' \overset{\mathtt{X}}{\to} \mathtt{M}$ with

$$\mathtt{X} = \mathtt{RightDivide}(\mathtt{G}, \mathtt{A}, \mathtt{N}),$$

once we have shown that the triple $(\mathtt{M}', \mathtt{X}, \mathtt{M})$ is compatible, i.e., that $\mathtt{M} \geq_{\text{row}} \mathtt{M}'\mathtt{X}$: For this note that $\mathtt{N} \geq_{\text{row}} \mathtt{M}'\mathtt{G}$ since γ is a morphism. Furthermore $\mathtt{N} \geq_{\text{row}} \mathtt{M}'\mathtt{YN}$, tautologically. Hence $\mathtt{N} \geq_{\text{row}} \mathtt{M}'\mathtt{G} - \mathtt{M}'\mathtt{YN} = \mathtt{M}'\mathtt{XA} = (\mathtt{M}'\mathtt{X})\mathtt{A}$. Now recall that φ is mono means that the kernel mono vanishes by Exercise 2.2.1, which translates to $\mathtt{M} \geq_{\text{row}} \mathtt{K} := \mathtt{SyzygiesOfRows}(\mathtt{A}, \mathtt{N})$ (cf. the proof of Proposition 3.2.11). But $\mathtt{K}$ row-dominates by definition any $\mathtt{T}$ with $\mathtt{N} \geq_{\text{row}} \mathtt{TA}$, in particular $\mathtt{M} \geq_{\text{row}} \mathtt{K} \geq_{\text{row}} \mathtt{M}'\mathtt{X}$.

To show that any epi $\varphi : \mathtt{M} \overset{\mathtt{A}}{\twoheadrightarrow} \mathtt{N}$ is a cokernel epi of its kernel mono $\kappa := (\ker\varphi, \mathtt{K}, \mathtt{M})$ (cf. Corollary 2.2.9) it suffices to show that unique colift $\varepsilon\backslash\varphi = (\operatorname{coker}\kappa = \left(\begin{smallmatrix}\mathtt{K}\\\mathtt{M}\end{smallmatrix}\right), \mathtt{A}, \mathtt{N})$ of φ along the cokernel epi $\varepsilon = (\mathtt{M}, \mathtt{1}, \left(\begin{smallmatrix}\mathtt{K}\\\mathtt{M}\end{smallmatrix}\right) = \operatorname{coker}\varphi)$ of κ is an isomorphism, where $\mathtt{K} := \mathtt{SyzygiesOfRows}(\mathtt{A}, \mathtt{N})$ (cf. the proof of Proposition 3.2.11):



An inverse morphism α of the colift $\varepsilon\backslash\varphi$ must satisfy $\alpha(\varepsilon\backslash\varphi) = 1_{\mathtt{N}}$. This implies for α that

$$\alpha := (\varepsilon\backslash\varphi)^{-1} : \mathtt{N} \overset{\mathtt{Y}}{\to} \left(\begin{smallmatrix}\mathtt{K}\\\mathtt{M}\end{smallmatrix}\right) = \operatorname{coker}\kappa \quad \text{with} \quad \mathtt{Y} := \mathtt{RightDivide}(\mathtt{1}, \mathtt{A}, \mathtt{N}),$$

where $\mathtt{1}$ is the identity matrix having the number of columns of $\mathtt{N}$ as its dimension. The existence of the matrix $\mathtt{Y}$ is equivalent to our assumption that φ is an epi. Now we show that the triple $\alpha = (\mathtt{N}, \mathtt{Y}, \left(\begin{smallmatrix}\mathtt{K}\\\mathtt{M}\end{smallmatrix}\right))$ is compatible and hence defines a morphism. By definition of $\mathtt{Y}$ there exists a matrix $\mathtt{Z}$ such that

$$(*) \qquad \mathtt{YA} + \mathtt{ZN} = \mathtt{1}.$$

Multiplying with $\mathtt{N}$ from the left we obtain $\mathtt{NYA} + \mathtt{NZN} = \mathtt{N}$, or equivalently $(\mathtt{NY})\mathtt{A} = (\mathtt{1} - \mathtt{NZ})\mathtt{N}$, i.e., $\mathtt{N} \geq_{\text{row}} (\mathtt{NY})\mathtt{A}$. But $\mathtt{K}$ row-dominates by definition any $\mathtt{T}$ with $\mathtt{N} \geq_{\text{row}} \mathtt{TA}$, hence $\mathtt{K} \geq_{\text{row}} \mathtt{NY}$ and α is a compatible triple. Since $\alpha(\varepsilon\backslash\varphi) = 1_{\mathtt{N}}$ by definition of $\mathtt{Y}$ it remains to show that $(\varepsilon\backslash\varphi)\alpha = 1_{\operatorname{coker}\kappa}$. To show this we multiply (*) with $\mathtt{A}$ from the left we obtain $\mathtt{AYA} + \mathtt{AZN} = \mathtt{A}$, or equivalently $(\mathtt{AY} - \mathtt{1})\mathtt{A} = (\mathtt{AZ})\mathtt{N}$, hence $\mathtt{N} \geq_{\text{row}} (\mathtt{AY} - \mathtt{1})\mathtt{A}$ and as above $\mathtt{K} \geq_{\text{row}} (\mathtt{AY} - \mathtt{1})$ and we are done. $\square$

[15] Lifts along monos are unique.

In the next chapter we will develop enough language to show that $R\text{-fpres}$ is equivalent to the category $R\text{-fpmod}$ of finitely presented modules over R. When R is left computable this will prove that the category of finitely presented R-modules is constructively ABELian.

CHAPTER 4

Functors, natural transformations, and adjunctions

In this chapter we will study functors as "1-morphism" between categories. It turns out that there are "2-morphisms" which we will call natural transformations; they are "morphisms between the functors". Finally, lectures on homological algebra without the notion of adjunction should be forbidden! End lecture 7

4.1. Functors

Definition 4.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **(covariant) functor** $F : \mathcal{A} \to \mathcal{B}$ consists of two maps $\mathcal{A}_0 \to \mathcal{B}_0$ (on objects) and $\mathcal{A}_1 \to \mathcal{B}_1$ (on morphisms[1]) such that

- $F(\operatorname{Hom}_{\mathcal{A}}(M, N)) \subset \operatorname{Hom}_{\mathcal{B}}(F(M), F(N))$;
- $F(\varphi\psi) = F(\varphi)F(\psi)$;
- $F(1_M) = 1_{F(M)}$

for all $M, N \in \mathcal{A}$ and any pair of composable $\mathcal{A}$-morphisms φ, ψ. We write $F \in \mathcal{B}^{\mathcal{A}}$.

We will see later how to turn $\mathcal{B}^{\mathcal{A}}$ into a category of functors.

The composition of functors is a functor and the operation is associative. The notions of **identity functor**, **invertible functor**, and **inverse functor** of an invertible functor are evident. However, invertible functors offer a far too restrictive notion of equivalence, which does not take the relative nature of categories into account, i.e., the possibility of many isomorphic objects to exist.

Definition 4.1.2. A functor $F : \mathcal{A} \to \mathcal{B}$ and consider the maps $F_{M,N} : \operatorname{Hom}_{\mathcal{A}}(M, N) \to \operatorname{Hom}_{\mathcal{B}}(F(M), F(N))$. F is called

- **faithful** $F_{M,N}$ is injective for all $M, N \in \mathcal{A}$.
- **full** $F_{M,N}$ is surjective for all $M, N \in \mathcal{A}$.
- **fully faithful** if F is full and faithful.

F is called **essentially surjective** (or **dense**) if for each of $L \in \mathcal{B}$ there exists an object $M \in \mathcal{A}$ such that $F(M) \cong L$ (in $\mathcal{B}$).

We start with some trivial examples.

Example 4.1.3.

(a) Viewing a monoid as a one-object category a functor is nothing but a homomorphism of monoids. The same is true for groups and one-object groupoids. Faithful functors are monos, full functors are epis, and invertible functors are isomorphisms.
(b) Functors between posets[2] are nothing but order-preserving maps.
(c) The **embedding functor** of a subcategory $\mathcal{A}' \subset \mathcal{A}$ is a faitful functor. $\mathcal{A}'$ is a full subcategory iff the embedding functor is full.
(d) The path components functor π_0 from (Top) to (Sets).
(e) The first fundamental group π_1 is a functor from (Top_0) to (Grps).
(f) The so-called **"forgetful" (or "underlying") functors** which forget some structure of the category
    - from graded S-modules to S-modules (where S is a graded ring)
    - from R-Mod to (Ab);
    - from (Ab) to (Sets_0);
    - from (Sets_0) to (Sets);

[1]You cannot speak of a functor if you don't know how to define it on morphisms!

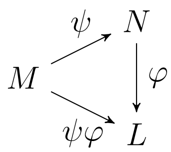
[2]viewed as skeletally thin categories 2.1.4.(1)

- from (Top) to (Sets);
- ...

(g) The **"free" functors** which associate to a set the "free" object on this set, e.g.,
- from (Sets) to (Ab);
- from (Sets) to R-Mod;
- ...

(h) ...

Main example 4.1.4. Let $\mathcal{A}$ be a category. Each object $M \in \mathcal{A}$ gives rise to a functor to (Sets), namely the **covariant** Hom**-functor**



$$h^M := \mathrm{Hom}_{\mathcal{A}}(M,-): \quad N \mapsto \mathrm{Hom}_{\mathcal{A}}(M,N),$$
$$(\varphi : N \to L) \mapsto \left(\mathrm{Hom}(M,\varphi) : \begin{cases} \mathrm{Hom}_{\mathcal{A}}(M,N) & \to \mathrm{Hom}_{\mathcal{A}}(M,L) \\ \psi & \mapsto \psi\varphi \end{cases} \right).$$

Exercise 4.1. Let $\mathcal{A}$ be a category. Construct from $\mathcal{A}$ so-called **arrow categories** in which $\ker$, coker, im, coim, pull-backs, push-outs (if they exist in $\mathcal{A}$) are functors.

Definition 4.1.5. A **contravariant functor** G between the categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. We write $G \in \mathcal{B}^{\mathcal{A}^{\mathrm{op}}}$. Since GROTHENDIECK we call them **presheaves on $\mathcal{A}$ with values on $\mathcal{B}$**.

The following example is one motivation behind calling contravariant functors "presheaves".

Example 4.1.6. The topology $\mathcal{T}$ of a topological space $(X,\mathcal{T})$ is a poset with the usual inclusions as morphisms, and thus a skeletally thin category 2.1.4.(1). One can associate to each open set $U \in \mathcal{T}$ the $\mathbb{R}$-vector space $\mathcal{O}_X(U)$ of continuous real valued functions on U and to each inclusion $U \subset V$ the restriction $\mathrm{res}^V_U : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$. The contravariant functor $\mathcal{O}_X : \mathcal{T} \to \mathbb{R}$-Vect is called the **structure sheaf** of the topological space X.

There are obvious analogous constructions for X a complex manifold with the EUCLIDian topology or an algebraic variety with ZARISKI topology, yielding the structure sheaf of holomorphic functions or regular functions, respectively.

In turned out that the ZARISKI topology is not rich enough to prove the WEIL conjectures. GROTHENDIECK, inspired by an idea of SERRE, introduced more general topologies called **sites**, which are allowed to include open sets "outside of X".

Example 4.1.7. Let $\mathcal{A}$ be a category and Δ the simplicial category. A presheaf in $\mathcal{A}^{\Delta^{\mathrm{op}}}$ is called a **simplicial object**. They are among the most powerful tools in modern algebraic topology.

Exercise 4.2. Describe how a **simplicial complex** can be fully encoded by a **simplicial set**[3] in $(\mathrm{Sets})^{\Delta^{\mathrm{op}}}$.

Exercise 4.3. Let $\mathcal{A}$ be a (small) category and $\mathcal{A}_n$ the class of sequences of length n. In particular, $\mathcal{A}_0$ is the set of objects (or identity morphisms), $\mathcal{A}_1$ the set of morphisms, and $\mathcal{A}_2 = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1$ the codomain of the multiplication map μ. The nerve $N(\mathcal{A})$ of a category $\mathcal{A}$ is the sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ together with specific morphisms $\mathcal{A}_i \to \mathcal{A}_j$ (including the four structure maps $1, s, t, \mu$) which should be constructed in this exercise. Show how $N(\mathcal{A})$ can become a simplicial set by specifying the missing induced structure morphisms. In this sense, a simplicial set is a vast generalization of a category[4].

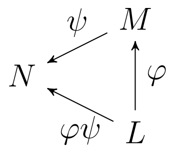
[3] However, in an a priori inefficient way.

[4] But thanks God, every simplicial set can again be modeled by a category :-)

The nerve of a group, regarded as a one object groupoid, will give us a standard resolution to compute group cohomology.

Exercise 4.4. The category of functors $\mathcal{A}^{\mathcal{B}}$ from a small $\mathcal{B}$ to an ABELian category $\mathcal{A}$ is again ABELian.

Main example 4.1.8. Let $\mathcal{A}$ be a category. Each object $N \in \mathcal{A}$ gives rise to a presheaf to (Sets), namely the **contravariant** Hom**-functor**



$$h_N := \mathrm{Hom}_{\mathcal{A}}(-, N) : \quad M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, N),$$
$$(\varphi : L \to M) \mapsto \left(\mathrm{Hom}(\varphi, N) : \begin{cases} \mathrm{Hom}_{\mathcal{A}}(M, N) & \to \mathrm{Hom}_{\mathcal{A}}(L, N), \\ \psi & \mapsto \varphi\psi \end{cases} \right).$$

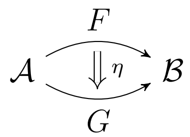
4.2. Natural transformations

Natural transformations will play the role of second level morphisms, i.e., the morphisms between the functors.

Definition 4.2.1. A **natural transformation** $\eta : F \to G$ of the functors $F, G : \mathcal{A} \to \mathcal{B}$ is a map $\eta : \mathcal{A}_0 \to \mathcal{B}_1$ such that the following diagram commutes

$$\begin{array}{ccc} F(M) & \xrightarrow{F(\varphi)} & F(N) \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ G(M) & \xrightarrow{G(\varphi)} & G(N) \end{array}$$

for all $\mathcal{A}$-morphisms $\varphi : M \to N$. η_M is called the **component** of η at $M \in \mathcal{A}$. We will usually write $\eta : F \implies G$ or $F \overset{\eta}{\implies} G$, but my favorite notation is the (less economic) globular one



as it emphasizes the 2-morphism nature of natural transformations. We denote the set of natural transformation between F and G by $[F, G]$ or $\mathcal{B}^{\mathcal{A}}(F, G)$.

The natural transformation η is called a **natural mono**, **natural epi**, or **natural isomorphism** if η_M is mono, epi, or an isomorphism for all $M \in \mathcal{A}$, respectively. If $F = G$ we then use the notions **natural endomorphisms** and **natural automorphisms** instead.

The functors F and G are called **equivalent** if there exists a natural isomorphism $\eta : F \implies G$. We write $F \simeq G$ or $F \simeq_\eta G$.

Example 4.2.2. Let $\mathcal{A}, \mathcal{B}$ be categories and $F, G, H \in \mathcal{B}^{\mathcal{A}}$.

(a) We now show that $\mathcal{B}^{\mathcal{A}}$ with functors as objects and natural transformations as morphisms build a category:
  - The map $1_F : \mathcal{A}_0 \to \mathcal{B}_1, M \mapsto 1_{F(M)}$ is a natural automorphism, called the **identity transformation** of F.
  - The source of a natural transformation $F \overset{\eta}{\implies} G$ is F and the target is G.
  - The composition of two natural transformations $F \overset{\eta}{\implies} G \overset{\tau}{\implies} H$ is defined in the obvious way $(\eta\tau)_M = \eta_M \tau_M$, i.e., by composing the two commutative diagrams

$$\begin{array}{ccc} F(M) & \xrightarrow{F(\varphi)} & F(N) \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ G(M) & \xrightarrow{G(\varphi)} & G(N) \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ H(M) & \xrightarrow{G(\varphi)} & H(N) \end{array}$$

and is again a natural transformation $F \overset{\eta\tau}{\Longrightarrow} H$. The composition of natural transformations is associative since it is defined by the composition of morphisms.

(b) The evaluation map $\varepsilon_V : V \to V^{**}, v \mapsto (\varphi \mapsto \varphi(v))$ between a k-vector space and its double dual is a natural mono between the identity functor $\mathrm{Id}_{k\text{-Vect}}$ and double-dualizing functor $D^2 : k\text{-Vect} \to k\text{-Vect}$. ε becomes a natural isomorphism if we restrict to the category of *finite* dimensional k-vector spaces. ε is also definable for R-Mod and R-mod,
End where it is in general neither mono nor epi.
lecture 8 (c) Using Exercise 4.1 we see that the kernel mono, cokernel epi, the coimage mono and epi, the image mono and epi, and the induced morphism $\overline{\varphi} : \mathrm{coim}\,\varphi \to \mathrm{im}\,\varphi$ are all natural transformation. In ABELian categories the latter is a natural isomorphism between the coimage and the image functors.

(d) Let R, S, T, U be rings and ${}_RM_S$ an (R,S)-bimodule[5]. Then the **tensor product** ${}_RM_S \otimes_S {}_SN_T$ (over S) is an (R,T)-bimodule for any (S,T)-bimodule ${}_SN_T$. Furthermore, the Hom-set $\mathrm{Hom}_R({}_RM_S, {}_RL_U)$ is an (S,U)-bimodule for any (R,U)-bimodule ${}_RL_U$.
The **left CURRYing isomorphism** of Hom-sets of left R-maps

$$\mathrm{Hom}_R({}_RM_S \otimes_S {}_SN_T, {}_RL_U) \cong \mathrm{Hom}_S({}_SN_T, \mathrm{Hom}_R({}_RM_S, {}_RL_U))$$

can be interpreted as a *tri*-natural isomorphism of (T,U)-bimodules.
Likewise, the **right CURRYing isomorphism** of Hom-sets of right T-maps

$$\mathrm{Hom}_T({}_RM_S \otimes_S {}_SN_T, {}_UK_T) \cong \mathrm{Hom}_S({}_RM_S, \mathrm{Hom}_T({}_SN_T, {}_UK_T))$$

can be interpreted as a *tri*-natural isomorphism of (U,R)-bimodules.

(e) An important special case of the CURRYing isomorphism is the following: Any morphism of rings $\varphi : S \to R$ induces via $ras := ra\varphi(s)$ an obvious (R,S)-bimodule structure on the ring R, which we then denote by ${}_RR_S$. For a left R-module L define the left **base-changed S-module**

$$_{\varphi}L := \mathrm{Hom}_R({}_RR_S, {}_RL).$$

Likewise, for a left S-module N define the left **base-changed R-module**

$$^{\varphi}N := {}_RR_S \otimes_S {}_SN.$$

The CURRYing isomorphism yields the **base-change formula** (for $T, U = \mathbb{Z}$)

$$\mathrm{Hom}_R({}^{\varphi}N, L) \cong \mathrm{Hom}_S(N, {}_{\varphi}L).$$

(f) **FROBENIUS reciprocity** is a special case of the base-change formula when φ is the inclusion of the subring $S \leq R$:

$$\mathrm{Hom}_R(R \otimes_S N, L) \cong \mathrm{Hom}_S(N, L_{|S}).$$

REMARK 4.2.3. If we view categories as objects and functors as morphisms we get a category of categories, with the restrictive notion of equality of functors. If we want to consider functors up to equivalence we get what is called a 2-category with the natural transformations as 2-morphisms. This process can be iterated indefinitely with marvelous applications in advanced mathematics.

Definition 4.2.4. Let $\mathcal{A}$ be a category.

- A functor $F : \mathcal{A} \to (\mathrm{Sets})$ is called **representable** if there exists an object $M \in \mathcal{A}$ such that $F \simeq h^M$. M is unique up to equivalence and we say that M represents F.

[5] I.e., a left R-module and a right S-module, simultaneously.

- A presheaf $G : \mathcal{A}^{\mathrm{op}} \to (\mathrm{Sets})$ is called **representable** if there exists an object $N \in \mathcal{A}$ such that $G \simeq h_N$. N is unique up to equivalence and we say that N represents G.

Example 4.2.5. The left CURRYing isomorphism (cf. 4.2.2.(d)) states that

- ${}_RM_S \otimes_S {}_SN_T$ represents the functor $\mathrm{Hom}_S({}_SN_T, \mathrm{Hom}_R({}_RM_S, -))$;
- $\mathrm{Hom}_R({}_RM_S, {}_RL_U)$ represents the presheaf $\mathrm{Hom}_R({}_RM_S \otimes_S -, {}_RL_U)$.

Lemma 4.2.6 (YONEDA lemma). *Let $\mathcal{A}$ be a category. For each object $M \in \mathcal{A}$ and each functor $F : \mathcal{A} \to (Sets)$ the map*

$$[h^M, F] \to F(M), (\eta : h^M \to F) \mapsto \eta_M(1_M) \in F(M)$$

is an isomorphism of sets. Likewise $[h_N, G] \cong G(N)$ for a presheaf $G : \mathcal{A}^{\mathrm{op}} \to (Sets)$.

PROOF. Exercise! □

As a corollary we obtain the YONEDA embedding.

Theorem 4.2.7 (YONEDA embedding). *Let $\mathcal{A}$ be a small category. The contravariant* Hom-*functor*

$$h_\bullet : \mathcal{A} \to (Sets)^{\mathcal{A}^{\mathrm{op}}}$$

is a full embedding *of $\mathcal{A}$ into the category of its set-valued presheaves.*

The usefulness of the YONEDA embedding comes from the fact, that much more categorical constructions (limits and colimits) exist in the category $(\mathrm{Sets})^{\mathcal{A}^{\mathrm{op}}}$, which we view as a sort of completion of $\mathcal{A}$. A constructed presheaf $F \in (\mathrm{Sets})^{\mathcal{A}^{\mathrm{op}}}$ "lies" in $\mathcal{A}$ iff F is representable. **MITCHEL's Embedding Theorem** of small ABELian category into a module category relies, of course, on the YONEDA embedding.

Now define notion of equivalence of categories.

Definition 4.2.8. An **equivalence of the categories** $\mathcal{A}$ and $\mathcal{B}$ is a pair of functors $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ with two natural isomorphisms $G \circ F \simeq \mathrm{Id}_{\mathcal{A}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{B}}$. The two categories $\mathcal{A}$, $\mathcal{B}$ are then called equivalent and we then write $F : \mathcal{A} \simeq \mathcal{B} : G$, or simply $\mathcal{A} \simeq \mathcal{B}$.

If $F : \mathcal{A} \simeq \mathcal{B} : G$ then the composed map $\mathrm{Hom}_{\mathcal{A}}(M, N) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(F(M), F(N)) \xrightarrow{G} \mathrm{Hom}_{\mathcal{A}}((G\circ F)(M), (G\circ F)(N)), \varphi \mapsto (G\circ F)(\varphi)$ is a bijection, given by conjugating with (the components of) the natural isomorphism. Hence the map $F : \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{B}}(F(M), F(N))$ is a bijection. In other words, F is fully faithful.

Now, for $B \in \mathcal{B}$ the B-component $B \cong F(G(B))$ of the natural isomorphism $F \circ G \simeq \mathrm{Id}_{\mathcal{B}}$ states that F is essentially surjective. Thus, we have proved one direct of the following criterion.

Proposition 4.2.9. *A functor $F : \mathcal{A} \to \mathcal{B}$ is part of an equivalence iff it is fully faithful and essentially surjective.*

PROOF. The other direction is immediate with a sufficiently strong axiom of choice. □

REMARK 4.2.10. Equivalence of categories preserves all **categorical properties**, i.e.,

- having initial, terminal, or zero objects;
- having kernels, cokernels, pull-backs, push-outs, products, coproducts;
- being pre-additive, additive, pre-ABELian, or ABELian.
- ...

In fact, one can always improve the above notion of equivalence to an "adjoint equivalence", which depends on the notion of adjoint functors.

Now we prove that R-fpres is indeed equivalent to the category of finitely presented R-modules. We already know that $\mathrm{Hom}_R(R^{1\times r}, R^{1\times c}) \cong R^{r\times c}$, i.e., that R-matrices are in bijection with R-maps between free modules of finite rank.

Definition 4.2.11. Let R be a ring. An left R-module N is called **finitely presented** if there exists a matrix $\mathtt{N} \in R^{r_N \times g_N}$ such that $N \cong \mathrm{coker}(R^{1\times r_N} \xrightarrow{\mathtt{N}} R^{1\times g_N})$. We denote the category of finitely presented modules by R-fpmod. It is subcategory of R-mod.

The images $(\overline{\mathtt{f}}_1, \ldots, \overline{\mathtt{f}}_{g_N})$ of the standard free generators $(\mathtt{f}_1, \ldots, \mathtt{f}_{g_N})$ of the free module $R^{1\times g_N}$ under the cokernel epi $\varepsilon_{\mathtt{N}} : R^{1\times g_N} \twoheadrightarrow N$ is a set of **generators** of N. By definition, $\mathtt{N}'\varepsilon_{\mathtt{N}} = 0_N$ iff $\mathtt{N} \geq_{\text{row}} \mathtt{N}'$, i.e., there exists a matrix $\mathtt{X}$ such that $\mathtt{N}' = \mathtt{XN}$.

The rows of the matrix $\mathtt{N}$ are the images of the standard free generators $(\mathtt{f}'_1, \ldots, \mathtt{f}'_{r_N})$ of the free module $R^{1\times r_N}$ under the R-map $\mathtt{N} : R^{1\times r_N} \to R^{1\times g_N}$. They are called a set of **relations** of N.

We now want to describe R-maps between $M = \operatorname{coker} \mathtt{M}$ and $N = \operatorname{coker} \mathtt{N}$. Let $\mathtt{A} \in R^{g_M \times g_N}$ be a matrix for which $\mathtt{N} \geq_{\text{row}} \mathtt{MA}$, i.e., for which there exists an $\mathtt{X}$ satisfying $\mathtt{MA} = \mathtt{XN}$.

$$\begin{array}{ccccc} R^{1\times r_M} & \xrightarrow{\mathtt{M}} & R^{1\times g_M} & \xrightarrow{\varepsilon_{\mathtt{M}}} & M \\ \downarrow{\scriptstyle \mathtt{X}} & & \downarrow{\scriptstyle \mathtt{A}} & & \downarrow{\scriptstyle \varphi_{\mathtt{A}}} \\ R^{1\times r_N} & \xrightarrow{\mathtt{N}} & R^{1\times g_N} & \xrightarrow{\varepsilon_{\mathtt{N}}} & N \end{array}$$

Such an $\mathtt{A}$ induces an R-map $\varphi = \varphi_{\mathtt{A}} : M \to N, \mathtt{m}\varepsilon_{\mathtt{M}} \mapsto \mathtt{mA}\varepsilon_{\mathtt{N}}$ forcing the right square to commute: $\varepsilon_{\mathtt{M}}\varphi = \mathtt{A}\varepsilon_{\mathtt{N}}$. We only need to see that φ is well-defined. For this we have to check that $\mathtt{mA}\varepsilon_{\mathtt{N}} = 0_N$ whenever $\mathtt{m}\varepsilon_{\mathtt{M}}$ vanishes. But $\mathtt{m}\varepsilon_{\mathtt{M}} = 0_M$ means that there exists a matrix $\mathtt{x} \in R^{1\times r_M}$ such that $\mathtt{m} = \mathtt{xM}$. Hence, $\mathtt{mA}\varepsilon_{\mathtt{N}} = \mathtt{xMA}\varepsilon_{\mathtt{N}} = \mathtt{xXN}\varepsilon_{\mathtt{N}} = 0_N$. Two matrices $\mathtt{A}, \mathtt{A}'$ give rise to same R-map φ iff $\mathtt{A}\varepsilon_{\mathtt{N}} = \mathtt{A}'\varepsilon_{\mathtt{N}}$, or equivalently, iff $(\mathtt{A} - \mathtt{A}')\varepsilon_{\mathtt{N}} = 0_N$ and hence iff $\mathtt{N} \geq_{\text{row}} \mathtt{A} - \mathtt{A}'$.

Conversely, an R-map $\varphi : M \to N$ defines a coefficients matrix $\mathtt{A}$ by $\overline{\mathtt{e}}_i\varphi = \mathtt{e}_i\varepsilon_{\mathtt{M}}\varphi = \sum_{j=1}^{g_N} \mathtt{A}_{ij}\overline{\mathtt{f}}_j$, where $(\mathtt{e}_1, \ldots, \mathtt{e}_{g_M})$ are the standard free generators of $R^{1\times g_M}$. So, by definition, $\varepsilon_{\mathtt{M}}\varphi = \mathtt{A}\varepsilon_{\mathtt{N}}$ and the right square commutes.

Summing up, we have just proved the following:

Theorem 4.2.12. *The essentially surjective functor*

$$\operatorname{coker} : \begin{cases} R\text{-fpres} & \to R\text{-fpmod}, \\ \mathtt{M} & \mapsto M = \operatorname{coker} \mathtt{M}, \\ (\mathtt{M}, \mathtt{A}, \mathtt{N}) & \mapsto \varphi_{\mathtt{A}} : M \to N = \operatorname{coker} \mathtt{N} \end{cases}$$

is fully faithful and is hence an equivalence of categories R-fpres $\simeq$ R-fpmod. If R is left computable then both categories are constructively ABEL*ian.*

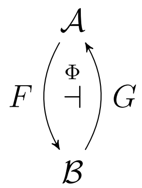
End lecture 9

4.3. Adjoint functors

Definition 4.3.1. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be two functors. The functor F is called **left adjoint** to a functor G (and G **right adjoint** to F) if there exists a *bi*natural isomorphism

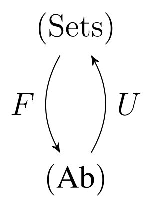
$$\Phi : \operatorname{Hom}_{\mathcal{B}}(F(-_{\mathcal{A}}), -_{\mathcal{B}}) \implies \operatorname{Hom}_{\mathcal{A}}(-_{\mathcal{A}}, G(-_{\mathcal{B}})).$$

We call Φ a Hom-adjunction between F and G and write $\Phi : F \dashv G$, $F \overset{\Phi}{\dashv} G$, or simply $F \dashv G : \mathcal{B} \to \mathcal{A}$. My favorite notation is again the globular one



Example 4.3.2.

(a) In the context of free and forgetful functors 4.1.3.(g),(f), e.g.,



we have a binatural isomorphism $\operatorname{Hom}_{(\text{Ab})}(F(S), N) \cong \operatorname{Hom}_{(\text{Sets})}(S, U(N))$.

(b) In the context of CURRYing isomorphism 4.2.2.(d): Each (R, S)-bimodule ${}_RM_S$ defines a pair of adjoint functors

$$ {}_RM_S \otimes_S - : S\text{-Mod} \rightleftarrows R\text{-Mod} : \mathrm{Hom}_R({}_RM_S, -). $$

Likewise, each (S, T)-bimodule ${}_SN_T$ defines a pair of adjoint functors

$$ - \otimes_S {}_SN_T : \text{Mod-}S \rightleftarrows \text{Mod-}T : \mathrm{Hom}_T({}_SN_T, -). $$

(c) In the context of FROBENIUS reciprocity 4.2.2.(f): Each subring $S \leq R$ gives rise to an adjunction between the inductions and the restriction functors

$$ \mathrm{ind}_S^R := R \otimes_S - : S\text{-Mod} \rightleftarrows R\text{-Mod} : {}_{|S} =: \mathrm{res}_S^R . $$

The proof of the following remark is not difficult but would at least require introducing the notion of (co)limit. For the tautological proof of special two cases see Example 5.3.4.

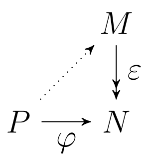
REMARK 4.3.3. One of the most decisive consequences of adjointness is that right adjoint functors are **continuous**, i.e., preserve (existing) limits, e.g., products and kernels. Dually, left adjoint functors (e.g., the above tensor product functors) are **co-continuous**, i.e., preserve (existing) colimits, e.g., coproducts and cokernels.

CHAPTER 5

Resolutions and derived functors

5.1. Projectives, injectives, and resolutions

Definition 5.1.1. Let $\mathcal{A}$ be a category. An object P in a category $\mathcal{A}$ is called **projective** if for each morphism $\varphi : P \to N$ and for each epi $\varepsilon : M \twoheadrightarrow N$



there exists a morphism $P \to M$, called a **projective lift (of** φ **along** ε making the above diagram commutes. The category $\mathcal{A}$ is said to have **enough projectives** if for any $L \in \mathcal{A}$ there exists an epi $P \twoheadrightarrow L$ with a projective P.

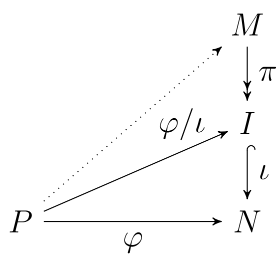
The dual concept is that of an **injective** object, **injective colift**, and **enough injectives**.

REMARK 5.1.2. Another way of phrasing this is to say:

- P is projective iff the covariant Hom-functor $\mathrm{Hom}(P, -) : \mathcal{A} \to (\mathrm{Sets})$ preserves epis.
- I is injective iff the contravariant Hom-functor $\mathrm{Hom}(-, I) : \mathcal{A}^{\mathrm{op}} \to (\mathrm{Sets})$ preserves monos.

REMARK 5.1.3. A supposedly "more general" form of the projective lift is often used in ABEL*ian* categories. The assumption of $\varepsilon : M \to N$ being epic can be relaxed by requiring that the image of $\varepsilon : M \to N$ dominates that of $\varphi : P \to N$.

PROOF. The existence of a **combined lift** $P \to M$ of φ along ε now follows from Prop. 2.2.8: There exists an essentially unique decomposition $M \overset{\pi}{\twoheadrightarrow} I \overset{\iota}{\hookrightarrow} N$ of ε into an epic π and a monic ι.



First take the lift φ/ι of (φ along ι), then the projective lift $P \to M$ of φ/ι along π. □

Proposition 5.1.4. *Let R be a ring.*

(a) A free module F in R-Mod or R-mod is projective.
(b) The categories R-Mod and R-mod has enough projectives.

PROOF.

(a) Let F be a free R-module, $\varphi : F \to N$ an R-map and $\pi : M \twoheadrightarrow N$ and R-epi (i.e., surjective). Let G be a free basis of F. For $g \in G$ define the projective lift $\widetilde{\varphi} : P \to M$ by mapping g to an arbitrary (but fixed) preimage of $g\varphi$ along π. The freeness of F implies that this a well-defined R-map which makes the above diagram commute.
(b) Let L be an R-module with generating set G (take a finite one in the case R-mod). Define F to be the free (hence projective) R-module on G, i.e., $F := \bigoplus_{g \in G} R$ with a free generating set G with $|G'| = |G|$. A bijection from G' onto G defines (uniquely) an R-epi $F \to L$.

□

Exercise 5.1. Let R be a ring and k a field.

(a) $\mathbb{Q}/\mathbb{Z}$ is an injective object in (Ab).
(b) R-Mod has enough injectives.
(c) $\mathbb{Z}-\mathrm{mod}$ and $k[x]-\mathrm{mod}$ do not have enough injectives.
(d) In $k[x]/\langle x^2\rangle$-mod the notions of free, projective, and injective coincide.
(e) $k[x,x^{-1}]$ is injective in the category $k[x]-\mathrm{grmod}$ of finitely generated graded $k[x]$-modules.

Proposition 5.1.5. *Let R be a left computable ring.*

(a) A free object $\mathtt{0} \in R^{0\times c}$ *in R-fpres is projective.*
(b) R-fpres has enough projectives.

PROOF.

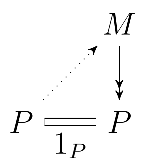
(a) Let $\mathtt{0} \xrightarrow{\mathtt{B}} \mathtt{N}$ be a morphism and $\mathtt{M} \xrightarrow{\mathtt{A}} \mathtt{N}$ be a epi in R-fpres. Then $\mathtt{0} \xrightarrow{\mathtt{X}} \mathtt{M}$ is a projective lift for $\mathtt{X} := \mathtt{RightDivide}(\mathtt{B},\mathtt{A})$.
(b) For $\mathtt{L} \in R^{?\times g_{\mathtt{L}}}$ define the free (hence projective) object $\mathtt{0} \in R^{0\times g_{\mathtt{L}}}$ and the epi $\mathtt{0} \xrightarrow{1} \mathtt{L}$.

□

Proposition 5.1.6. *Let R be a ring and P an R-module. The following are equivalent:*

(a) P is projective.
(b) Any epi onto P is split.
(c) P is a summand of a free module.

PROOF.
(a) $\Longrightarrow$ (b) Let $M \twoheadrightarrow P$ be an epi. A section is given by a projective lift in the diagram:



(b) $\Longrightarrow$ (c) Take an epi π from a free R-module F onto P (see proof of Proposition 5.1.4.(b)). Choose a split $\sigma : P \hookrightarrow F$. Hence: $F = \ker\pi \oplus \underbrace{\operatorname{im}\sigma}_{\cong P}$.

(c) $\Longrightarrow$ (a) Let P be a direct summand of a free R-module $F = P' \oplus P$. Every R-map $P \to N$ can be extended to an R-map $F \to N$ by mapping the direct summand P' to zero. Since F is projective we can construct projective lifts on F and then restrict them to P. □

Example 5.1.7.

(a) $\mathbb{Z}/2\mathbb{Z}$ is neither projective in $\mathbb{Z}-\mathrm{mod}$ nor in $\mathbb{Z}/4\mathbb{Z}-\mathrm{mod}$.
(b) $\mathbb{Z}/2\mathbb{Z}$ is projective in $\mathbb{Z}/2\mathbb{Z}-\mathrm{mod}$ and $\mathbb{Z}/6\mathbb{Z}-\mathrm{mod}$.

PROOF.

(a) The two epis $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}, 1 \mapsto 1+2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}, 1+4\mathbb{Z} \mapsto 1+2\mathbb{Z}$ are not split.
(b) $\mathbb{Z}/2\mathbb{Z}$ is a free in $\mathbb{Z}/2\mathbb{Z}$-module and a direct summand of the free $\mathbb{Z}/6\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

□

Proposition 5.1.8. *Let R be a principal ideal* domain. *For a finitely generated module M*

$$M \text{ is torsion-free} \iff M \text{ is projective} \iff M \text{ is free.}$$

PROOF. The SMITH normal form states that any finitely generated module M is of the form $R^r \oplus R/\langle d_1\rangle \oplus \cdots \oplus R/\langle d_\ell\rangle$ with $0 \neq d_1 \mid d_2 \mid \cdots \mid d_\ell$. For M to be torsion-free (or projective or free) the torsion part $\bigoplus_{i=1}^{\ell} R/\langle d_i\rangle$ must vanish. So we see that M torsion-free module is equivalent to projective and to free. □

Theorem 5.1.9. *A finitely generated projective module over a commutative local ring is free.*

PROOF. Let $\mathfrak{m}$ be the maximal ideal of R and $k = R/\mathfrak{m}$ the residue class field of R. Let P be projective R-module and r the minimal number of generators of P, i.e., the dimension of $P/\mathfrak{m}P \cong k \otimes_R P$. Hence P is a direct summand of $R^r = P \oplus Z$. Tensoring with k yields the vector space $k^r = R^r/\mathfrak{m}R^r = P/\mathfrak{m}P \oplus Z/\mathfrak{m}Z \cong k^r \oplus k^a$, which implies that $a = 0$. Hence $Z = 0$ by NAKAYAMA's Lemma. □

The statement of the theorem is true for any module (finitely generated or not) over a not necessarily commutative local ring. This was proved by KAPLANSKY in 1958.

Theorem 5.1.10 (QUILLEN-SUSLIN Theorem)**.** *Let k a field or $k = \mathbb{Z}$. A finitely generated projective module over $k[x_1, \dots, x_n]$ is free.*

There is a constructive proof of this theorem for constructive fields but it is too long to treat it here. See (`https://github.com/homalg-project/LessGenerators/blob/master/gap/QuillenSuslin.gi`) for an implementation.

Definition 5.1.11. Let $\mathcal{A}$ be ABELian category and $M \in A$. A **projective resolution** of M is a (possibly infinite) sequence

$$P_\bullet : P_0 \xleftarrow{\partial_1} P_1 \xleftarrow{\partial_2} P_2 \xleftarrow{\partial 3} \cdots$$

of projective objects P_i together with an epi $M \overset{\pi}{\twoheadleftarrow} P_0$ such that the **augmented projective resolution** $M \twoheadleftarrow P_\bullet$ is an exact sequence.

The kernel $K_1 := \ker \pi \overset{\mu_1}{\hookrightarrow} P_0$ is called the 1**-st syzygy object** of M. More generally, one can define the n**-th syzygy object**[1] K_n to be the kernel of ∂_{n-1} (with $\partial_0 := \pi$).

The projective resolution is said to be **finite** of **length** n if n is the smallest integer such that $K_{n+1} = 0$, otherwise **infinite**. The minimal length (or ∞) of a projective resolution is called the **projective dimension** of M. In particular, the projective dimension is zero iff M is projective.

The dual notion is that of an **injective resolution**, **cosyzygy object**, and **injective dimension**.

If $\mathcal{A}$ is R-Mod or R-mod then a **free resolution** is a projective resolution where the objects P_i are all free.

REMARK 5.1.12. Let $\mathcal{A}$ be an ABELian category with enough projectives resp. injectives then each object M admits a (possibly infinite) projective resp. injective resolution. If $\mathcal{A}$ is constructively ABELian then the above is true constructively.

Example 5.1.13. Let k be a field.

(a) The exact sequence

$$k \twoheadleftarrow k[x,y,z]^{1\times 1} \xleftarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} k[x,y,z]^{1\times 3} \xleftarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}} k[x,y,z]^{1\times 3} \overset{\begin{pmatrix} x & y & z \end{pmatrix}}{\hookleftarrow} k[x,y,z]^{1\times 1}$$

is an augmented free resolution of $k \cong k[x,y,z]/\langle x,y,z\rangle$ of length 3. This is known in 3-dimensional vector analysis as: $\operatorname{div} \circ \operatorname{rot} = \nabla(\nabla \times \cdot) = 0$ and $\operatorname{rot} \circ \operatorname{grad} = \nabla \times (\nabla \cdot) = 0$.

(b) Let R be a principal ideal *domain*, e.g., $R = \mathbb{Z}$ or $R = k[x]$. The HERMITE normal form tells us that any finitely generated R-module has a resolution of length 1: $M \twoheadleftarrow R^{1\times c} \overset{\mathtt{H}}{\hookleftarrow} R^{1\times r}$. In particular, any module is at most of projective dimension 1.

(c) Let A is a commutative *domain*, $f, g \in A \setminus \{0, 1\}$ and $R = A/\langle fg\rangle$. Then the 2-periodic sequence $R \xleftarrow{f} R \xleftarrow{g} R \xleftarrow{f} \cdots$ is an infinite free (and hence projective) resolution of $R/\langle f\rangle$ in $R-$mod and there is no finite *free* resolution. If f, g are coprime, i.e., if $\langle f, g\rangle = A$, then $R \cong R/\langle f\rangle \oplus R/\langle g\rangle$ and the mono $R \overset{f}{\hookleftarrow} R/\langle g\rangle$ is a length 1 projective (non free) resolution of $R/\langle f\rangle$ in $R-$mod.

(c.1) The periodic sequence $\mathbb{Z}/4\mathbb{Z} \xleftarrow{2} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2} \mathbb{Z}/4\mathbb{Z} \xleftarrow{2} \cdots$ is an infinite free resolution of $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}-$mod. $\mathbb{Z}/2\mathbb{Z}$ has no finite projective resolution in $\mathbb{Z}/4\mathbb{Z}-$mod.

[1] All these objects are defined up to so-called **projective equivalence**. This follows from SCHANUEL's Lemma, but we won't need this fact.

(c.2) The mono $\mathbb{Z}/6\mathbb{Z} \overset{2}{\hookleftarrow} \mathbb{Z}/3\mathbb{Z}$ is a length 1 projective (non free) resolution of $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/6\mathbb{Z}-\mathrm{mod}$. The 2-periodic sequence $\mathbb{Z}/6\mathbb{Z} \overset{2}{\leftarrow} \mathbb{Z}/6\mathbb{Z} \overset{3}{\leftarrow} \mathbb{Z}/6\mathbb{Z} \overset{2}{\leftarrow} \mathbb{Z}/6\mathbb{Z} \overset{3}{\leftarrow} \cdots$ is an infinite free resolution of $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/6\mathbb{Z}-\mathrm{mod}$.

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(c.3) Consider the integral group ring $\mathbb{Z}[C_n] \cong \mathbb{Z}[x]/\langle x^n - 1\rangle$ of the cyclic group C_n. The 2-periodic sequence

$$\mathbb{Z}[x]/\langle x^n - 1\rangle \xleftarrow{x-1} \mathbb{Z}[x]/\langle x^n - 1\rangle \xleftarrow{x^{n-1}+x^{n-2}+\cdots+1} \mathbb{Z}[x]/\langle x^n - 1\rangle \xleftarrow{x-1} \cdots$$

is an infinite free resolution of the trivial C_n-module $\mathbb{Z} \cong \mathbb{Z}[x]/\langle x-1\rangle$. This will prove that finite cyclic groups has non-vanishing integral group (co)homology in arbitrary high (co)dimensions.

(c.4) Consider the integral group ring $\mathbb{Z}[C_\infty] \cong \mathbb{Z}[x, x^{-1}]$ of the infinite cyclic group $C_\infty \cong (\mathbb{Z}, +)$. The mono $\mathbb{Z}[x, x^{-1}] \overset{x-1}{\hookleftarrow} \mathbb{Z}[x, x^{-1}]$ is a length 1 free resolution of the trivial C_∞-module $\mathbb{Z} \cong \mathbb{Z}[x]/\langle x-1\rangle$. It is the shortest possible free (even projective) resolution. This will prove that the integral group (co)homology of the infinite cyclic group will vanish in (co)dimensions higher than 1.

Exercise 5.2. Consider the integral group ring $\mathbb{Z}[C_\infty^n] \cong \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ of the free ABELian group $C_\infty^n \cong (\mathbb{Z}^n, +)$ of rank n. Construct a finite free resolution of the trivial C_∞^n-module $\mathbb{Z} \cong \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]/\langle x_1 - 1, \ldots, x_n - 1\rangle$ of length n.

5.2. Homotopies of chain morphisms

In this section we want to introduce an important equivalence relation among cochain morphisms and hence a new isomorphism notion between complexes: "the homotopy equivalence".

Definition 5.2.1. Let $\mathcal{A}$ be an additive category. Two chain morphism

$$\mu_\bullet, \nu_\bullet : (M_\bullet, \partial_\bullet^M) \to (N_\bullet, \partial_\bullet^N)$$

in $\mathcal{A}$ are called **homotopic** if there exists degree $+1$ morphism $h : M_\bullet \to N_\bullet$ such that[2] $\mu_\bullet - \nu_\bullet = \partial_\bullet^M h_\bullet + h_\bullet \partial_\bullet^N$. We then call $h_\bullet$ a **chain homotopy** from $\mu_\bullet$ to $\nu_\bullet$ and write[3] $h_\bullet : \mu_\bullet \implies \nu_\bullet$.

A chain morphism homotopic to the zero complex is called **null-homotopic** (or **homotopic to zero**). In other words, two chain morphisms are homotopic if there difference is null-homotopic. A complex for which the identity chain morphism is null-homotopic is called **contractible**.

This defines an equivalence relation among chain morphisms which does not only respect the additive structure but defines an **(two-sided) ideal** in $\mathrm{Ch}_? \mathcal{A}$, i.e., closed under left and right composition. This is an easy exercise.

The factor category of

$$K_?\mathcal{A} := \mathrm{Ch}_? \mathcal{A}/\sim_{\text{chain homotopy}} \qquad ? \in \{\bullet, \geq 0, +, \leq 0, -\}$$

is called the **homotopy category** of the additive category $\mathcal{A}$. Two chain complexes $M_\bullet, N_\bullet \in \mathrm{Ch}_? \mathcal{A}$ are called **homotopy equivalent** if they are isomorphic in $K_?\mathcal{A}$, i.e., if there exists chain morphisms $\mu_\bullet : M_\bullet \rightleftarrows N_\bullet : \nu_\bullet$ such that $\mu_\bullet\nu_\bullet$ is homotopic to $1_\bullet^M : M_\bullet \to M_\bullet$ and $\nu_\bullet\mu_\bullet$ is homotopic to $1_\bullet^N : N_\bullet \to N_\bullet$.

The homotopy category of an additive category is additive. It is a basic example of a so-called **triangulated category**.

From now on we are in the context of an ABELian category $\mathcal{A}$.

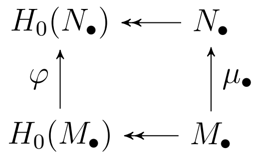
This Proposition will be the first ingredient for proving that derived functors are well-defined, despite their choice-dependent definition.

Proposition 5.2.2. *All projective resolutions of an object are homotopy equivalent.*

[2]The plus sign in $\partial_\bullet^M h_\bullet + h_\bullet \partial_\bullet^N$ will only become important in the subsequent proofs.

[3]So $-h_\bullet : \nu_\bullet \implies \mu_\bullet$

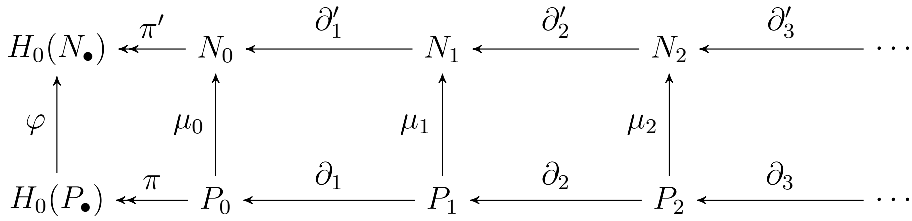
For the proof we need some preparation. We start with the following terminology. Let $M_\bullet, N_\bullet \in \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ be two (equally indexed) complexes and $\varphi : H_0(M_\bullet) \to H_0(N_\bullet)$ a morphism. In this context a **lift** of φ is a chain morphism $\mu_\bullet : M_\bullet \to N_\bullet$ which makes the augmented diagram commute



The following proposition is a general useful result.

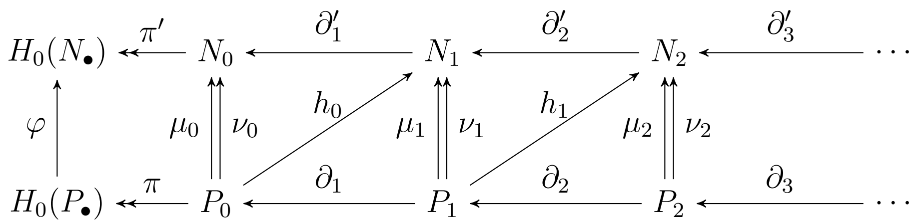
Proposition 5.2.3. *Let $N_\bullet \in \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ be an acyclic complex and $P_\bullet$ be an equally indexed complex with projective objects. Then any morphism $\varphi : H_0(P_\bullet) \to H_0(N_\bullet)$ lifts to a chain morphism $\mu_\bullet : P_\bullet \to N_\bullet$ and all such lifts are homotopic.*

PROOF. We will repeatedly apply Remark 5.1.3.



We start by constructing μ_0 as the projective lift of $\pi\varphi$ along the epi π'. Since $\partial_1\mu_0\pi' = \underbrace{\partial_1\pi}_{=0}\varphi = 0$ the definition of $H_0(N_\bullet)$ implies that the image of ∂'_1 dominates that of $\partial_1\mu_0$ and we can construct $\mu_1 : P_1 \to N_1$ as a combined lift. Now we proceed by induction: Since $\partial_{i+1}\mu_i\partial'_i = \underbrace{\partial_{i+1}\partial_i}_{=0}\mu_{i-1}$ the acyclicity of $N_\bullet$ implies that the image of ∂'_{i+1} dominates that of $\partial_{i+1}\mu_i$ and we set $\mu_{i+1} : P_{i+1} \to N_{i+1}$ as a combined lift.

Now let $\nu_\bullet$ be another lift of φ.



Since $(\nu_0 - \mu_0)\pi' = 0$ the image of ∂'_1 dominates that of $\nu_0 - \mu_0$ and there exists a combined lift $h_0 : P_0 \to N_1$ satisfying $\nu_0 - \mu_0 = h_0\partial'_1$. To construct $h_1 : P_1 \to N_2$ as a combined lift which satisfies the desired homotopy identity we need show that the image of $\nu_1 - \mu_1 - \partial_1 h_0$ is dominated by that of ∂'_2. To check this we post-compose with ∂'_1:

$$(\nu_1 - \mu_1 - \partial_1 h_0)\partial'_1 = \partial_1(\nu_0 - \mu_0) - \partial_1 h_0 \partial'_1 = 0$$

by the above identity. Now we iterate. □

PROOF OF PROPOSITION 5.2.2. Let $P_\bullet$ and $P'_\bullet$ be two projective resolutions of $M \cong H_0(P_\bullet) \cong H_0(P'_\bullet)$. Then by Proposition 5.2.3 there exists lifts $\mu_\bullet : P_\bullet \to P'_\bullet$ and $\mu'_\bullet : P'_\bullet \to P_\bullet$ of 1_M. We conclude that $\mu_\bullet\mu'_\bullet : P_\bullet \to P_\bullet$ is a lift of 1_M, but so is $1_{P_\bullet} : P_\bullet \to P_\bullet$ and hence they are homotopic. The same holds for the reverse composition $\mu'_\bullet\mu_\bullet$ and we are done. □

The second and final ingredient for proving that derived functors are well-defined is the following proposition.

Proposition 5.2.4. *Homotopic chain morphisms induce* equal *morphisms on homology.*

PROOF. The chain morphism $\partial_\bullet^M h_\bullet + h_\bullet \partial_\bullet^N$ obviously sends $Z(M_\bullet)$ into $B(N_\bullet)$. Hence it induces the zero morphism $H_i(M_\bullet) \to H_i(N_\bullet)$ for all $i \in \mathbb{Z}$. □

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5.3. Additive, left exact, and right exact functors

The minimal requirement for functors we want to derive here is the additivity.

Definition 5.3.1. A functor $F : \mathcal{A} \to \mathcal{B}$ of pre-additive categories is called **additive** if it induces homomorphisms $F_{M,N} : \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{B}}(F(M), F(N))$.

Exercise 5.3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of additive categories. Then the following are equivalent:

- F is additive.
- F preserves biproducts (in particular, $F(M \oplus N) \cong F(M) \oplus F(N)$).

Hint: Use Remark 2.2.4.

Example 5.3.2.

(a) Let $\mathcal{A}$ be a pre-additive category. Unravelling the definition of the Hom-functor shows that $\mathrm{Hom} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to (\mathrm{Ab})$ is additive in each argument.

(b) Left adjoints and right adjoints of additive functors of pre-additive categories are additive. This is an easy exercise.

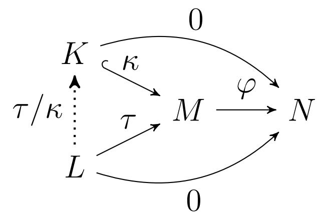
(c) The tensor product functors (see Example 4.3.2.(b)) are additive (e.g., as left adjoints of covariant Hom-functors).

Although we will apply the notion of derivation to additive functors the functors we consider will usually have an the additional property of left or right exactness.

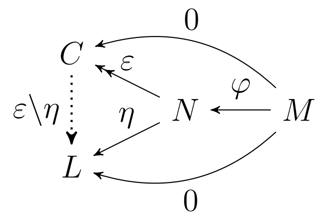
Definition 5.3.3. A functor $F : \mathcal{A} \to \mathcal{B}$ between ABELian categories is called **left exact** if it preserves kernels[4]. It is called **right exact** if it preserves cokernels[5]. A contravariant functor $\mathcal{A} \to \mathcal{B}$ is treated as a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Example 5.3.4. Let $\mathcal{A}$ be an additive category and $L \in \mathcal{A}$.

(a) The covariant Hom-functor $\mathrm{Hom}_{\mathcal{A}}(L, -)$ is left exact. This is simply another way of expressing the universal property of a kernel (even in an arbitrary category with zero!)



(b) Dually, the *contra*variant Hom-functor $\mathrm{Hom}_{\mathcal{A}}(-, L)$ is *also* left exact (not right exact!) Again, this is another way of expressing the universal property of a cokernel (even in an arbitrary category with zero!)



(c) As we mentioned in Remark 4.3.3 the tensor product functors are right exact, as they are left adjoint functors.

[4]or, equivalently, short left exact sequences.

[5]or, equivalently, short right exact sequences.

5.4. Derived functors of ABELian categories

As we have seen above, some very important functors between ABELian categories are merely left or right exact. There is a way to make such a functor exact by *throwing away information*, which is realized by passing to the "derived functor between the corresponding derived categories". Surprisingly, this process of "throwing away information" on the level of categories reveals a way to extract more information out of the non-exact functor, already on the level of ABELian categories.

Definition 5.4.1. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of ABELian categories.

(L) Suppose $\mathcal{A}$ has enough projectives. The **i-th left derived functor $\mathbf{L}_i F$** is defined *on objects* by

$$(\mathbf{L}_i F)(M) := H_i(F(P_\bullet^M)),$$

where $P_\bullet^M$ is any projective resolution of $M \in \mathcal{A}$.

(R) Suppose $\mathcal{A}$ has enough injectives. The **i-th right derived functor $\mathbf{R}^i F$** is defined *on objects* by

$$(\mathbf{R}^i F)(M) := H^i(F(I_M^\bullet)),$$

where $I_M^\bullet$ is any injective resolution of $M \in \mathcal{A}$.

To define the morphism part of these functors we only need to show how the homology and cohomology functors H_i and H^i act on morphisms. It is an immediate corollary of Propositions 5.2.4 and 5.2.2 that the above notions are well-defined. More precisely:

REMARK 5.4.2. If we consider the full additive subcategory of $\operatorname{Ch}_+ \mathcal{A}$ of complexes with projective objects and factor out the ideal of zero homotopic morphisms then we recover the derived category $\mathbf{D}_+\mathcal{A}$, a special kind of triangulated categories. The i-th left derived functors can be recovered as the composition of three functors $\mathbf{L}_i F := H_i \circ F \circ P$:

- the "embedding" functor $P : \mathcal{A} \hookrightarrow \mathbf{D}_+\mathcal{A}$ which associates to $M \in \mathcal{A}$ a projective resolution $P_\bullet^M$. The morphism part of this functor is defined by Proposition 5.2.3.
- the derived functor $\mathbf{L}F : \mathbf{D}_+\mathcal{A} \to \mathbf{D}_+\mathcal{B}$ which is simply F applied to complexes with projective objects and homotopy classes of chain morphisms. It is an exact functor of additive[6] categories.
- the **i-th homology functor** $H_i : \mathbf{D}_+\mathcal{B} \to \mathcal{B}, P_\bullet \to H_i(P_\bullet)$. The morphism part of this functor will be treated later on. The functor is well-defined by Proposition 5.2.4.

REMARK 5.4.3. The i-th left (resp. right) derived functor of a contravariant functor $G : \mathcal{A} \to \mathcal{B}$ is defined as the i-th left (resp. right) derived of the covariant functor $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. Hence, for $\mathbf{L}_i$ we use injective and for $\mathbf{R}^i$ projective resolutions in $\mathcal{A}$.

Proposition 5.4.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of ABELian categories (where $\mathcal{A}$ has enough projective or injective when necessary).*

(a) If F is right exact then $\mathbf{L}_0 F \simeq F$, and, dually,
(b) if F is left exact then $\mathbf{R}^0 F \simeq F$.

PROOF. Let $0 \leftarrow M \leftarrow P_0 \leftarrow P_1$ be the beginning of a projective resolution $P_\bullet$ of an $M \in \mathcal{A}$. If F preserves cokernels implies $0 \leftarrow F(M) \leftarrow F(P_0) \leftarrow (P_1)$ that $F(M) \cong H_0(F(P_\bullet))$. □

Proposition 5.4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, $G : \mathcal{A} \to \mathcal{B}$ a left exact contravariant functor, and assume that $\mathcal{A}$ has enough projectives. Let M an object in $\mathcal{A}$, $P_\bullet$ a projective resolution of M, and K_i the i-th syzygy object of M (cf. Definition 5.1.11), i.e.,*

$$M \twoheadleftarrow P_0 \leftarrow \cdots P_{i-1} \stackrel{\mu_i}{\hookleftarrow} K_i.$$

Then

- $(\mathbf{L}_i F)(M) \cong \ker F(\mu_i)$.
- $(\mathbf{R}^i F)(M) \cong \operatorname{coker} G(\mu_i)$.

[6]We only defined all notions of exactness for ABELian categories. They can be defined for general categories.

PROOF. We start by splitting $P_{i-1} \leftarrow P_i$ as the composition $P_{i-1} \overset{\mu_i}{\hookleftarrow} K_i \twoheadleftarrow P_i$. Since the right exact functor F respects epis we can replace P_i by K_i. Finally, the contravariant G recall that it sees the epi $K_i \twoheadleftarrow P_i$ as a mono! ☐

In fact, the previous proposition is the definition of **satellites**. So we have just proved that the satellites of left or right exact functors naturally coincide with the derived functors.

5.5. The definitions of Ext and Tor

Definition 5.5.1. Let $\mathcal{A}$ be an ABELian category. Define the i-th extension bifunctor $\mathrm{Ext}^i_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to (\mathrm{Ab})$ as follows:

- If $\mathcal{A}$ has enough projectives then set
$$\mathrm{Ext}^i_{\mathcal{A}}(M, N) := (\mathbf{R}^i \operatorname{Hom}_{\mathcal{A}}(-, N))(M).$$
- If $\mathcal{A}$ has enough injectives then set
$$\mathrm{Ext}^i_{\mathcal{A}}(M, N) := (\mathbf{R}^i \operatorname{Hom}_{\mathcal{A}}(M, -))(N).$$

If $\mathcal{A}$ has enough projectives and enough injectives then both definitions naturally coincide by a simple spectral sequence argument. In fact, there is a third definition, due to YONEDA, independent of the existence of enough projectives or injectives [**BLH14**, §4]. This more intrinsic definition is naturally isomorphic to each of the above definitions once any of them exists.

Definition 5.5.2. Let $\mathcal{A}$ be an ABELian category of modules R-Mod or, if R is a left NOETHERian ring, R-mod. Define the i-th torsion bifunctor $\mathrm{Tor}^R_i : \mathcal{A} \times \mathcal{A} \to (\mathrm{Ab})$ as follows:

$$\mathrm{Tor}^R_i(M, N) := (\mathbf{L}_i(- \otimes_R N))(M), \quad \text{or}$$
$$\mathrm{Tor}^R_i(M, N) := (\mathbf{L}_i(M \otimes_R -))(N).$$

Again, both definitions naturally coincide by a simple spectral sequence argument.

From Proposition 5.4.4 we deduce for the above context

Corollary 5.5.3.

- $\mathrm{Ext}^0_{\mathcal{A}} \simeq \mathrm{Hom}_{\mathcal{A}}$;
- $\mathrm{Tor}^R_0 \simeq \otimes_{\mathbb{R}}$.

5.6. Very first steps in group (co)homology

One of the many fruitful applications of homological algebra is the study of group (co)homology. Group cohomology had an early implicit appearance in HILBERT's work, most prominently in his famous theorem HILBERT 90. Non-ABELian group cohomology was introduced by group theorists under the name "Faktorensysteme" at the beginning of the 20th century. Here we mention early works of SCHUR and HOPF. The first book on homological algebra is that of CARTAN and EILENBERG [**CE99**]. It already treated among other things group homology and cohomology. Even ABELian categories appeared for the first time under the name "exact categories" in the appendix by BUCHSBAUM in loc. cit. However, it was GROTHENDIECK who, a couple of years later, was able to come up with a unified formalism for space and group (co)homology [**Gro57**]. The missing ingredient in [**CE99**] was that of a sheaf and its injective resolution.

For the rest of the section let G be a group, $\mathbb{Z}[G]$ its integral group ring, and M a (left) G-module. The G-module $\mathbb{Z}$ will always denote the trivial G-module[7].

Definition 5.6.1.

- The group homologies of G with values in M are defined by
$$H_i(G, M) := \mathrm{Tor}^{\mathbb{Z}[G]}_i(\mathbb{Z}, M).$$

[7] In GROTHENDIECK's language the G-module M is a G-equivariant sheaf and the trivial G-module $\mathbb{Z}$ is the constant sheaf $\underline{\mathbb{Z}}$ on the one point space.

- The group cohomologies of G with values in M are defined by
$$H^i(G,M) := \operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},M).$$

By corollary 5.5.3 the 0-th group homology and cohomology are easy to describe: End lecture 12

Proposition 5.6.2.

- *The 0-th group cohomology is naturally isomorphic to the G-submodule of G-**invariants***
$$\begin{aligned} H^0(G,M) &\cong \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M) \\ &\cong M^G := \{m \in M \mid gm = m \text{ for all } g \in G\}. \end{aligned}$$
It is the largest submodule of M on which G acts trivially.
- *The 0-th group homology is naturally isomorphic to the factor G-module of G-**coinvariants***
$$\begin{aligned} H_0(G,M) &\cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \\ &\cong M_G := M/\{m(g-1) \mid m \in M \text{ and } g \in G\}. \end{aligned}$$
It is the largest factor module of M on which G acts trivially.

PROOF.
- A G-equivariant map $\varphi : \mathbb{Z} \to M$ is defined by $\varphi(1)$ which must be a fixed point of the G-action.
- Consider the short exact sequence $\mathbb{Z} \twoheadleftarrow \mathbb{Z}[G] \hookleftarrow IG$ where IG is the augmentation ideal of G. Note that IG is a free $\mathbb{Z}$-module on the free basis $B_G := \{g-1 \mid g \in G\}$. By the right exactness of the tensor product we deduce that $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ is the cokernel of the map $(\mathbb{Z}[G] \hookleftarrow IG) \otimes_{\mathbb{Z}[G]} M = M \leftarrow IG \otimes_{\mathbb{Z}[G]} M$. The image of the last map is $(IG)M$. □

For the next step we need Proposition 5.4.5.

Proposition 5.6.3. $H_1(G,\mathbb{Z}) \cong G^{\mathrm{ab}} := G/G'$.

PROOF. We compute $H_1(G,\mathbb{Z})$ as the kernel of the map $(\mathbb{Z}[G] \hookleftarrow IG) \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}[G] \leftarrow IG \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ which is the zero map since G acts trivially. First we prove that $IG \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong IG/IG^2$. Note that $IG \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is isomorphic to $IG \otimes_{\mathbb{Z}} \mathbb{Z} \cong IG$ modulo the ABELian subgroup generated by elements of the form $(g-1)h \otimes z - (g-1) \otimes \underbrace{hz}_{=z} = (g-1)(h-1) \otimes z$. This proves the first claim. Now we prove that $IG/IG^2 \cong G/G'$. The map $B_G := \{g-1 \mid g \in G\} \to G/G', g-1 \mapsto g$ extends uniquely to a map $IG \to G/G'$. IG^2 is in the kernel of this map since $(g-1)(h-1) = (gh-1)-(g-1)-(h-1)$. We obtain a map $IG/IG^2 \to G/G'$ which is inverted by mapping the class of g to $(g-1)+IG^2$. □

Exercise 5.4. Prove that $H^1(G,\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(G,\mathbb{Z}),\mathbb{Z})$.

For a proof see [**HS97**, p. 193].

In Example 5.1.13 we have encountered very simple 2-periodic free resolutions of a finite cyclic group. Hence all group (co)homologies of a finite cyclic group with coefficients in a trivial module are easy computable:

Corollary 5.6.4. *Let C_k be a finite cyclic group and M a trivial C_k-module (e.g., $\mathbb{Z}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{C}^*$, $\mathbb{R}$, $\mathbb{C}$, ...). Then*

- $H_0(C_k,M) \cong M$, $H_{2i-1}(C_k,M) \cong \operatorname{coker}(M \xrightarrow{k} M)$, *and* $H_{2i}(C_k,M) \cong \ker(M \xrightarrow{k} M)$ *for all* $i = 1,2,\dots$.
- $H^0(C_k,M) \cong M$, $H^{2i-1}(C_k,M) \cong \ker(M \xrightarrow{k} M)$, *and* $H^{2i}(C_k,M) \cong \operatorname{coker}(M \xrightarrow{k} M)$ *for all* $i = 1,2,\dots$.

CHAPTER 6

Generalized morphisms and functoriality

Diagram chasing is a standard technique in homological algebra to construct morphisms. However, this involves arguing with elements which are missing in general ABELian categories. A standard argument to circumvent this difficulty is to embed the *small* ABELian category into a category of modules $R\text{-Mod}$. The problem with this argument is that it breaks constructivity. In this chapter we introduce the new notion of a generalized morphism. The idea is simply to add new morphisms to an ABELian category until diagram chasing becomes a series of generalized lifts and compositions, and in particular, expressible by simple formulas.

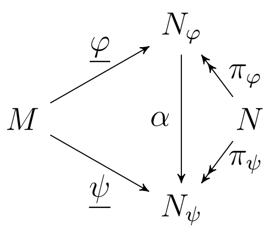
6.1. Generalized morphisms and their two arrow calculus

Definition 6.1.1. Let $\mathcal{A}$ be an ABELian category and $M, N \in \mathcal{A}$. A **generalized morphism** with **source** M and **target** N is an equivalence class of pairs of morphisms[1]

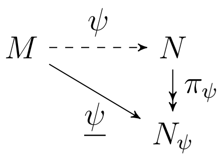
$$\psi := (M \stackrel{\underline{\psi}}{\longrightarrow} N_\psi \stackrel{\pi_\psi}{\twoheadleftarrow} N),$$

where two such paris are regarded equivalent if

- their sources and targets coincide
- and $\psi \sim \varphi := (M \stackrel{\underline{\varphi}}{\longrightarrow} N_\varphi \stackrel{\pi_\varphi}{\twoheadleftarrow} N)$ if there exists[2] an isomorphism $\alpha : N_\varphi \to N_\psi$ such that the following diagram commutes:



For such an equivalence class we write more elaborately



We call $\underline{\varphi}$ the morphism **associated** to φ and π_ψ the **morphism aid**[3] of ψ. We say that ψ is an **honest morphism** if its aid π_ψ is an isomorphism (in $\mathcal{A}$).

REMARK 6.1.2. An honest generalized morphism $\psi : (M \stackrel{\underline{\psi}}{\longrightarrow} N_\psi \stackrel{\pi_\psi}{\twoheadleftarrow} N)$ is identified with the morphism $\underline{\psi}/\pi_\psi$. The distinguished representative of the class ψ is $(M \stackrel{\underline{\psi}/\pi_\psi}{\longrightarrow} N \stackrel{1_N}{\twoheadleftarrow} N)$.

The image of a morphism in an ABELian category is, by definition, a subobject of the target, namely the one defined by the (class of monos mutually dominating the) image mono; cf. Definition 1.2.18 and Corollary 2.2.8. The true motivation behind introducing generalized morphisms is

[1] Since an epi $\pi_\psi : N \to N_\psi$ in an ABELian category mutually codominates its coimage epi $N \to N/\ker \pi_\psi$, both epis define the same factor object of N. In particular, $N_\psi \cong N/\ker \pi_\psi$.

[2] If such an isomorphism α exists then it is the unique colift $\pi_\varphi \backslash \pi_\psi$.

[3] As it "aids" ψ to become a morphism, namely $\underline{\psi}$.

to enlarge ABELian categories by a new sort of morphisms which have *subfactor* objects of their targets as “images”.

Definition 6.1.3. Let $\psi : M \xrightarrow{\underline{\psi}} N_\psi \overset{\pi_\psi}{\twoheadleftarrow} N$ be a generalized morphism in an ABELian category. We define the **generalized image** of ψ to be the image of $\underline{\psi}$ and write

$$\operatorname{gim}\psi := \operatorname{im}\underline{\psi}.$$

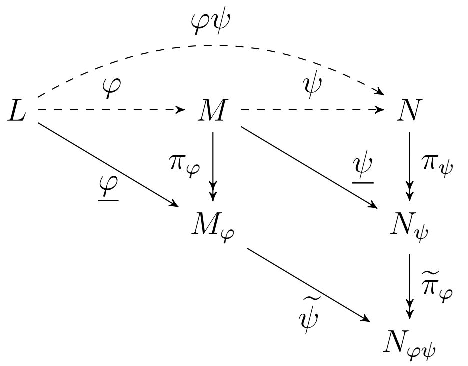
It is a subobject of N_ψ and thus a subfactor object of the target N.

Further we define the **combined image** as the pre-image

$$\operatorname{cim}\psi := \pi_\psi^{-1}(\operatorname{gim}\psi).$$

It is a subobject of the target N.

Definition 6.1.4. We define the **composition** of two generalized morphisms $\varphi : (L \xrightarrow{\underline{\varphi}} M_\varphi \overset{\pi_\varphi}{\twoheadleftarrow} M)$ and $\psi : (M \xrightarrow{\underline{\psi}} N_\psi \overset{\pi_\psi}{\twoheadleftarrow} N)$ by adding the push-out $N_{\varphi\psi}$ of π_φ and $\underline{\psi}$



$$\varphi\psi := (M \xrightarrow{\underline{\varphi}\widetilde{\psi}} N_{\varphi\psi} \overset{\pi_\psi\widetilde{\pi}_\varphi}{\twoheadleftarrow} N)$$

Proving the associativity of the above composition is a tedious exercise. We obtain the following result.

Theorem 6.1.5. *Let $\mathcal{A}$ be an* ABEL*ian category. The above composition turns the quiver* $\mathbf{G}\mathcal{A}$ *with* $(\mathbf{G}\mathcal{A})_0 = \mathcal{A}_0$ *and generalized morphisms as morphisms into a category, which naturally contains* $\mathcal{A}$ *as the subcategory of honest morphisms (and the same class of objects).*

Exercise 6.1. Let $\mathcal{A}$ be an ABELian category and $\varphi : (L \xrightarrow{\underline{\varphi}} M_\varphi \overset{\pi_\varphi}{\twoheadleftarrow} M)$ a generalized morphism of $\mathcal{A}$.

(a) The epis of $\mathbf{G}\mathcal{A}$ are those of $\mathcal{A}$.
(b) φ is mono in $\mathbf{G}\mathcal{A}$ iff $\underline{\varphi}$ is mono in $\mathcal{A}$.

Hint: For (a) consider the composition of φ with 1_M and $M \overset{\pi_\varphi}{\twoheadrightarrow} M_\varphi \overset{\pi_\varphi}{\twoheadleftarrow} M$, respectively.

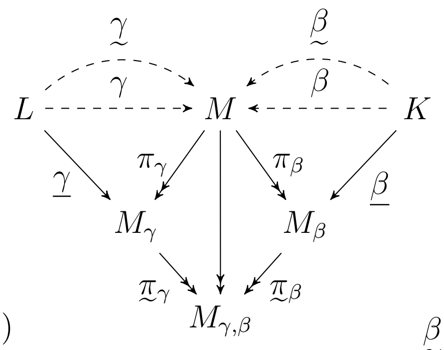
Although $\mathbf{G}\mathcal{A}$ has no zero object (not even an initial object) we can define kernels and cokernels as follows:
The kernel and cokernel of φ in $\mathbf{G}\mathcal{A}$ are those of $\underline{\varphi}$, where the cokernel epi of φ is the composition of π_φ with the cokernel epi of $\underline{\varphi}$ (viewed as morphisms in $\mathbf{G}\mathcal{A}$).

6.2. The generalized Lifting Lemma

In this section we prove a lifting lemma for generalized morphisms. It turns out that diagram chasing can be replaced by successive compositions and liftings of generalized morphisms. To this end we introduce some notation. We start by describing the general context of this section:

Let $\mathcal{A}$ be an ABELian category and $\beta : (K \xrightarrow{\underline{\beta}} M_\beta \overset{\pi_\beta}{\twoheadleftarrow} M)$ and $\gamma : (L \xrightarrow{\underline{\gamma}} M_\gamma \overset{\pi_\gamma}{\twoheadleftarrow} M)$ be two generalized morphisms with the same target M.

Definition 6.2.1. We define the **common coarsenings** $\underset{\sim}{\beta}$ and $\underset{\sim}{\gamma}$ of β and γ by adding a push-out diagram of π_β and π_γ



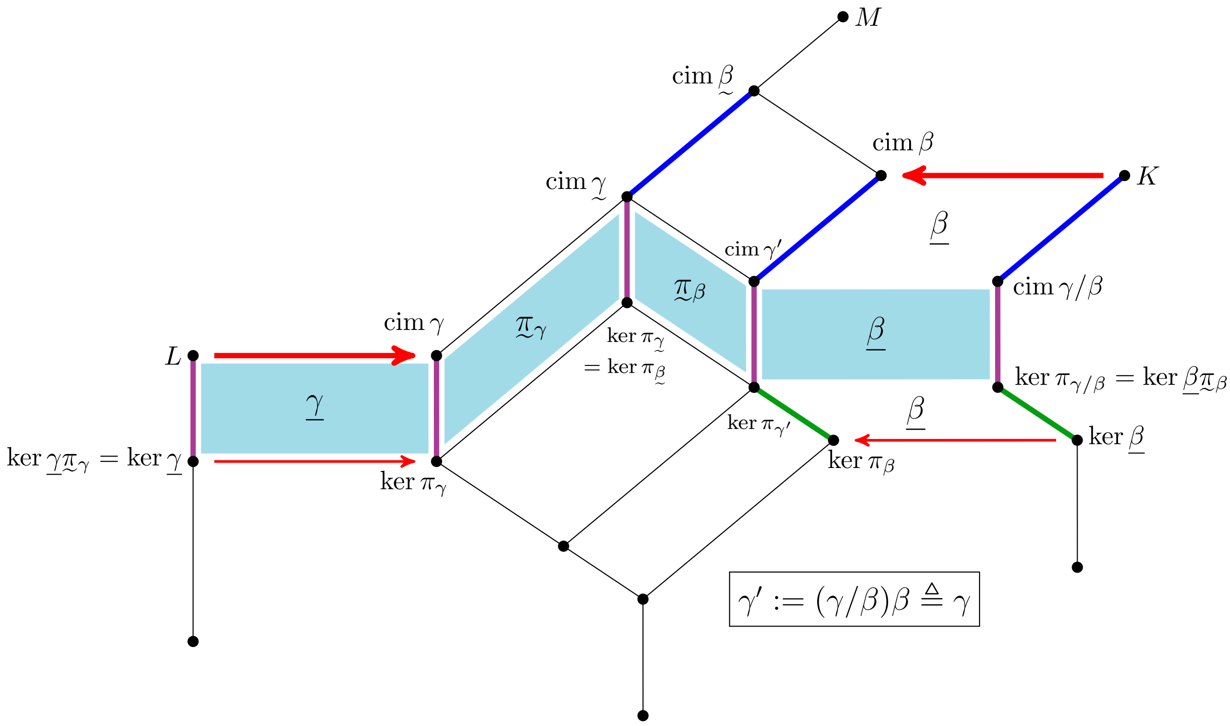
$$\underset{\sim}{\gamma} = (L \xrightarrow{\underline{\gamma}\underset{\sim}{\pi}_\gamma} M_{\gamma,\beta} \overset{\pi_\gamma \underset{\sim}{\pi}_\gamma}{\twoheadleftarrow} M) \qquad \underset{\sim}{\beta} = (K \xrightarrow{\underline{\beta}\underset{\sim}{\pi}_\beta} M_{\gamma,\beta} \overset{\pi_\beta \underset{\sim}{\pi}_\beta}{\twoheadleftarrow} M)$$

Note that $\pi_\beta \underset{\sim}{\pi}_\beta = \pi_\gamma \underset{\sim}{\pi}_\gamma$. We call $\underset{\sim}{\gamma}$ **the coarsening of** γ **with respect to** β and write $\gamma \underset{\beta}{\rightsquigarrow} \underset{\sim}{\gamma}$. We say that

- β **dominates** γ if the combined image of $\underset{\sim}{\beta}$ is larger than that of $\underset{\sim}{\gamma}$, or, equivalently, if the image of $\underline{\beta}\underset{\sim}{\pi}_\beta$ is larger than that of $\underline{\gamma}\underset{\sim}{\pi}_\gamma$ (cf. Definition 1.2.14);
- the coarsening $\gamma \underset{\beta}{\rightsquigarrow} \underset{\sim}{\gamma}$ is **effective** if $\underset{\sim}{\pi}_\gamma$ is a mono, or, equivalently, if $\ker \underset{\sim}{\gamma} = \ker \gamma$, i.e., if $\ker(\underline{\gamma}\underset{\sim}{\pi}_\gamma) = \ker \underline{\gamma}$;
- two generalized morphisms γ, γ' with equal source and target are **equal up to effective common coarsenings** or **quasi-equal** if their common coarsenings are equal *and* are *both* effective. We then write $\gamma \triangleq \gamma'$. If both γ and γ' are honest morphisms then $\triangleq$ reduces to the equality of morphisms in $\mathcal{A}$.

End lecture 13

We now define the generalized lift of two generalized morphisms with the same target under two conditions. I claim that generalized lifts together with compositions of generalized morphisms capture the whole complexity of functoriality and diagram chasing.



Definition 6.2.2. In the context of the previous definition we assume that

(`dom`) β dominates γ; and
(`eff`) the coarsening $\gamma \underset{\beta}{\rightsquigarrow} \underset{\sim}{\gamma}$ is effective.

The **generalized lift** of γ along β is the generalized morphism

$$\gamma/\beta : (L \xrightarrow{\underline{\gamma/\beta}} M_{\gamma/\beta} \overset{\pi_{\gamma/\beta}}{\twoheadleftarrow} K)$$

where

- $M_{\gamma/\beta}$ is the generalized image of $\underset{\sim}{\beta}$ (i.e., the image of $\underline{\beta}\underset{\sim}{\pi}_\beta$),
- $\pi_{\gamma/\beta}$ the image epi of $\underline{\beta}\underset{\sim}{\pi}_\beta$, and
- $\underline{\gamma/\beta}$ the co-restriction of $\underline{\gamma}\underset{\sim}{\pi}_\gamma$ to $M_{\gamma/\beta}$ (i.e., the lift of $\underline{\gamma}\underset{\sim}{\pi}_\gamma$ along the image mono of $\underline{\beta}\underset{\sim}{\pi}_\beta$).

This is the key Lemma of this lecture course.

Lemma 6.2.3 (Lifting Lemma)**.** *The generalized lift γ/β of γ along β satisfies*

$$(\gamma/\beta)\beta \triangleq \gamma.$$

If γ and β are honest morphisms then $\triangleq$ reduces to equality of morphisms in $\mathcal{A}$.

PROOF. Follow the shaded area from left two right! □

We will now start to demonstrate the power of these notions.

REMARK 6.2.4. Epis in $\mathbf{G}\mathcal{A}$ (these are exactly the epis in $\mathcal{A}$) are always split (in $\mathbf{G}\mathcal{A}$). We call the pre-inverse of $\varepsilon : M \twoheadrightarrow N$ defined by

$$\varepsilon^{-1} := 1_N/\varepsilon : (N \xrightarrow{1_N} N \overset{\varepsilon}{\twoheadleftarrow} M)$$

the **generalized pre-inverse** or **generalized section** of ε.

6.3. Generalized natural transformations and functoriality

The first real advantage behind these notions is that there are more generalized natural transformations than non-generalized ones.

Definition 6.3.1. A **generalized natural transformation** $\eta : F \to G$ of the additive functors $F, G : \mathcal{A} \to \mathcal{B}$ between ABELian categories is a map $\eta : \mathcal{A}_0 \to (\mathbf{G}\mathcal{B})_1$ such that the usual diagram commutes (in $\mathbf{G}\mathcal{A}$)

$$\begin{array}{ccc} F(M) & \xrightarrow{F(\varphi)} & F(N) \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ G(M) & \xrightarrow{G(\varphi)} & G(N) \end{array}$$

for all $\mathcal{A}$-morphisms $\varphi : M \to N$.

Let $\operatorname{Arr}\mathcal{A}$ denote the arrow category of $\mathcal{A}$ (cf. Exercise 4.1). Recall that the kernel mono κ is a natural mono

$$\operatorname{Arr}\mathcal{A} \underset{\ker}{\overset{s}{\rightrightarrows}} \mathcal{A}, \quad \Uparrow \kappa$$

Dually, the cokernel epi ε is a natural epi

$$\operatorname{Arr}\mathcal{A} \underset{\operatorname{coker}}{\overset{t}{\rightrightarrows}} \mathcal{A}, \quad \Downarrow \varepsilon$$

But now we can define a generalized natural mono which starts at coker by pre-inverting ε in $\mathbf{G}\mathcal{A}$:

$$\operatorname{Arr}\mathcal{A} \underset{\operatorname{coker}}{\overset{t}{\rightrightarrows}} \mathcal{A}, \quad \Uparrow \varepsilon^{-1}$$

In $\mathcal{A}$ we "relate" a subobject K of an object $M \in \mathcal{A}$ by a mono from K to M, and, dually, a factor object N of M by and epi in the reverse direction, i.e., from M to N. Now recall the definition of a subfactor object (Definition 2.2.11). There we considered two subobjects $L \overset{\tau}{\hookrightarrow} M \overset{\kappa}{\hookleftarrow} K$ of M with $L \leq K$. We already called the two morphisms

$$K/L := \operatorname{coker}\left(L \overset{\tau/\kappa}{\hookrightarrow} K\right) \hookrightarrow M/L \twoheadleftarrow M$$

a generalized mono! So using generalized morphisms we can relate a subfactors of $M \in \mathcal{A}$ (including subobjects and factor objects) by *generalized monos from the subfactors to M*. In particular, we get a generalized mono from the defect of exactness functor to the functor which picks the middle object of two composable $\mathcal{A}$-morphisms.

REMARK 6.3.2. More generally, we get a generalized mono $\eta^i : H^i(M_\bullet) \hookrightarrow M_i$. Let Id^i denote the additive functor from $\mathrm{Ch}^\bullet \mathcal{A}$ to $\mathcal{A}$ defined by $M^\bullet \mapsto M^i$, $\mu^\bullet \mapsto \mu^i$. Saying that η^i is a generalized natural mono

$$\mathrm{Ch}^\bullet \mathcal{A} \underset{H^i}{\overset{\mathrm{Id}^i}{\rightrightarrows}} \mathcal{A} \qquad (\Uparrow \eta^i)$$

defines the yet missing functoriality of the (co)homology H^i as a generalized lift

$$H^i(M^\bullet \xrightarrow{\mu^\bullet} N^\bullet) := (\eta^i_{M^\bullet}\mu^i)/\eta^i_{N^\bullet} : H^i(M^\bullet) \to H^i(N^\bullet).$$

In particular, the latter is an *honest* morphism. I leave the details as an exercise.

As we were able to define the functoriality of the (co)homology using a formula involving compositions and generalized lifts of generalized morphisms we can do the same for connecting homomorphisms.

Proposition 6.3.3. *Let $(R_\bullet, \partial^R_\bullet) \overset{\pi_\bullet}{\twoheadleftarrow} (C_\bullet, \partial_\bullet) \overset{\iota_\bullet}{\hookleftarrow} (A_\bullet, \partial^A_\bullet)$ be a short exact sequence of (not necessarily exact) complexes in an* ABEL*ian category. The i-**th connecting homomorphism** δ_i defined by combing*

$$H_{i-1}(A_\bullet) \xrightarrow{(\eta_{i-1})_{A_\bullet}} A_{i-1} \xrightarrow{\iota_{i-1}} C_{i-1} \xleftarrow{\partial_i} C_i \xrightarrow{\pi_i} R_i \xleftarrow{(\eta_i)_{R_\bullet}} H_i(R_\bullet)$$

$$(\widetilde{\eta}_{i-1})_{A_\bullet} := H_{i-1}(A_\bullet) \to C_{i-1}, \qquad (\widetilde{\eta}_i)_{R_\bullet} := H_i(R_\bullet) \to C_i$$

to

$$((\widetilde{\eta}_i)_{R_\bullet}\partial_i)/(\widetilde{\eta}_{i-1})_{A_\bullet} : H_i(R_\bullet) \to H_{i+1}(A_\bullet)$$

is an honest *morphism which yields the **long** **exact** **homology sequence***

$$\cdots \xleftarrow{\delta_{i-1}} H_i(R_\bullet) \xleftarrow{H_i(\iota_\bullet)} H_i(C_\bullet) \xleftarrow{H_i(\pi_\bullet)} H_i(A_\bullet) \xleftarrow{\delta_i} H_{i+1}(R_\bullet) \xleftarrow{H_{i+1}(\iota_\bullet)} H_{i+1}(C_\bullet) \xleftarrow{H_{i+1}(\pi_\bullet)} \cdots$$

6.4. Spectral sequences

An m-filtered complex is complex $C_\bullet$ together with a chain $0 \leq C^1_\bullet \leq \cdots C^{m-1}_\bullet \leq C^m_\bullet$. A short exact sequence of complexes is thus nothing but a 2-filtered complex $0 \leq A_\bullet \leq C_\bullet$. Long exact sequences of short exact sequences generalize to so-called spectral sequences of filtered complexes. One can prove that all involved higher differentials can be given by a closed formula merely involving compositions and generalized lifts of generalized morphisms.

Definition 6.4.1 (Filtered complex)**.** We distinguish between chain and cochain complexes:

(a) A chain of subcomplexes $(F_pC)_{p\in\mathbb{Z}}$ (i.e. $\partial(F_pC_n) \leq F_pC_{n-1}$ for all n) of the chain complex $(C_\bullet, \partial)$ is called an **ascending filtration** if $F_{p-1}C \leq F_pC$. The p-th **graded part** is the subfactor chain complex defined by $\operatorname{gr}_p C := F_pC/F_{p-1}C$.

(d) A chain of subcomplexes $(F^pC^n)_{p\in\mathbb{Z}}$ (i.e. $\partial(F^pC^n) \leq F^pC^{n+1}$ for all n) of the *co*chain complex $(C^\bullet, \partial)$ is called a **descending filtration** if $F^pC \geq F^{p+1}C$. The p-th **graded part** is the subfactor cochain complex defined by $\operatorname{gr}^p C := F^pC/F^{p+1}C$.

Like for modules all filtrations of complexes will be **exhaustive** (i.e. $\bigcup_p F_pC = C$), **HAUSDORFF** (i.e. $\bigcap_p F_pC = 0$), and will have **finite length** m (i.e. the difference between the highest and the lowest stable index is at most m). Such filtrations are called m-step filtrations in the sequel.

Convention: For our purposes filtrations on chain complexes are automatically ascending whereas on *co*chain complexes descending.

Definition 6.4.2 (Homological spectral sequence)**.** A **homological spectral sequence** (starting at r_0) in an ABELian category $\mathcal{A}$ consists of

(a) Objects $E^r_{pq} \in \mathcal{A}$, for $p, q, r \in \mathbb{Z}$ and $r \geq r_0 \in \mathbb{Z}$; arranged as a sequence (indexed by r) of lattices (indexed by p, q);
(b) Morphisms $\partial^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ with $\partial^r\partial^r = 0$, i.e. the sequences of slope $-\frac{r+1}{r}$ in E^r form a chain complex;
(c) Isomorphisms between E^{r+1}_{pq} and the homology $\ker \partial^r_{pq}/\operatorname{im} \partial^r_{p+r,q-r+1}$ of E^r at the spot (p, q).

E^r is called the r-th **sheet** (or **page**, or **term**) of the spectral sequence.

Note that E^{r+1}_{pq} is by definition (isomorphic to) a subfactor of E^r_{pq}. p is called the **filtration degree** and q the **complementary degree**. The sum $n = p + q$ is called the **total degree**. A morphism with source of total degree n, i.e. on the n-th diagonal, has target of degree $n-1$, i.e. on the $(n-1)$-st diagonal. So the total degree is *decreased* by one.

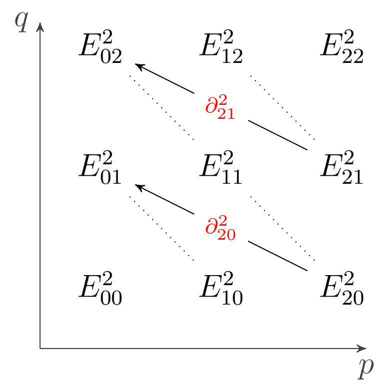
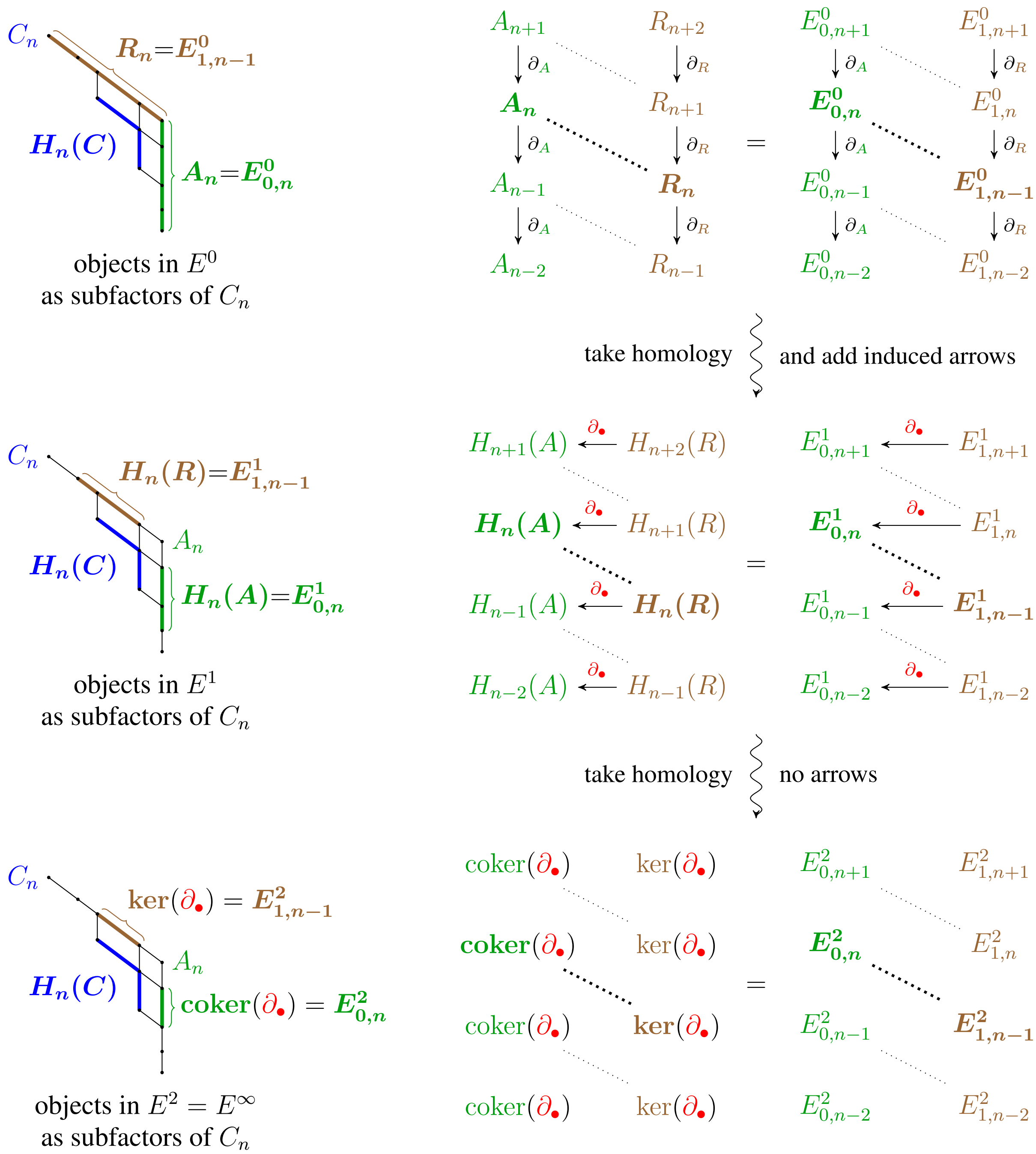


FIGURE 1. E^2

Part of the data we have in the context of long exact sequences can be put together to construct a spectral sequence with three pages E^0, E^1, and E^2:



with $p, q \in \mathbb{Z}$, $n = p + q$. Taking the two columns over $p = 0$ and $p = 1$, for example, is equivalent to setting $F_{-1}C := 0$, $F_0C := A$, and $F_1C := C$.

This can now be generalized to m-filtrations. To get the morphism $\partial^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ we use the analogous formula:

$$E^r_{p-r,q+r-1} \xrightarrow{(\eta^r_{p-r,q+r-1})_{C_\bullet}} C_{p+q-1} \xleftarrow{\partial_{p+q}} C_{p+q} \xleftarrow{(\eta^r_{pq})_{C_\bullet}} E^r_{pq}$$

$$\partial^r_{pq} := ((\eta^r_{pq})_{C_\bullet}\partial_{p+q})/(\eta^r_{p-r,q+r-1})_{C_\bullet}$$

End lecture 14

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